Essays in Macroeconomics and Finance

by

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A dissertation submitted to The Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy.

Baltimore, Maryland

September, 2017

Abstract

This dissertation consists of three papers. The first paper studies the comovement between returns to stocks and nominal Treasury bonds, which varies over time in both magnitude and direction. Earlier research attempts to interpret this phenomenon as a consequence of variations in the link between inflation and future economic activity. I present some opposing empirical evidence, and instead argue that in the data, the comovement between stock and nominal bond returns is driven by real factors. I build a New Keynesian model that generates this behavior through the joint dynamics of output, inflation, and interest rates. The model features two types of persistent shocks to productivity growth: mean-reverting cyclical shocks and permanent trend shocks. The relative importance of these two shocks varies stochastically over time. The model quantitatively explains the observed patterns in stock-bond return comovement.

The goal of the second paper is to quantify variation in the volatility of firm-level productivity shocks and study its impact via the accumulation of capital across firms. I first document robust empirical evidence on the upward trend in firm-level productivity shocks volatility. Then, I develop a tractable general equilibrium model to study the consequences of the increase in idiosyncratic volatility. The model features heterogeneous firms which make irreversible investment decisions over time.

The third papers investigates the cross-sectional pricing of idiosyncratic volatility risk by presenting a new model for idiosyncratic stock return volatility. In the model, idiosyncratic volatility consists of two components. One is a long-run component and could be modeled as containing a unit root. The other is short-run and is less persistent. Compared to models used in the literature, this model can better capture the persistence of idiosyncratic volatility in the long-run. Estimating the model using the cross-section of stock returns, I decompose the idiosyncratic volatility into short-run and long-run components and explore the cross-sectional pricing of different components. I find that there is a significantly negative relationship between expected long-run volatility and expected return. In contrast, expected short-run volatility is not found to be significantly related to expected return. These findings remain robust after controlling for return reversals.

Primary Reader: Professor Gregory Duffee

Secondary Reader: Professor Jon Faust

Acknowledgments

I'm sincerely grateful to my advisors Gregory Duffee, Jon Faust for their invaluable support and guidance throughout the Ph.D. program. I'm also thankful to my colleagues at the Johns Hopkins University for their friendship and support.

My deepest gratitude goes to my parents. This dissertation would not exist without for their dedication, encouragement and love.

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1.1 Introduction

The stock-bond return correlation is strongly time-varying. In particular, the sign of the correlation turned from positive to negative in the late 1990s. There is a growing literature documenting this time variation using sophisticated statistical models (see, e.g., Guidolin and Timmermann (2006)) but much less work attempting to disentangle its macroeconomic sources. These stylized facts raise the question of what macroeconomic forces determine the risk exposure of U.S. Treasury bonds, and in particular the time variation of risk.

Most papers in the literature such as David and Veronesi (2013), Campbell et al. (2014), Li (2011) and Hasseltoft (2009) focus on the correlation between stock and nominal bonds returns and attempt to explain this phenomenon through variations of the link between inflation and economic activity. This approach appears to be inconsistent with the empirical evidence reported in this paper.

I document novel empirical evidence that the correlation between stock returns and nominal bond returns is closely related to that between stock returns and real bond returns. By using data from both US and UK, I find that this changing pattern of correlation between stocks and bonds applies to both nominal and real bonds. During the mid 1990s, the stock-bond correlation was as high as 60 percent and by early 2000s it dropped to levels as low as −60 percent. What is more striking is that the correlation between stock returns and nominal bond returns move closely with the correlation between stock and real bond returns. The contribution of this paper is to add a time-varying real component to a New Keynesian model and show it can jointly account for the dynamics of output, interest rates, inflation, and importantly stock-bond return correlation.

The key mechanism of the model works through the cyclical and trend component of productivity growth. The cyclical component of productivity growth mean-reverts: a positive shock to productivity corresponds to lower expected consumption growth. Lower expected consumption growth translates into lower real interest rates and higher prices for bonds. Stock and bond returns are, therefore, positively correlated in response to cyclical shocks. The trend component of productivity growth contains a unit root. A positive shock to productivity corresponds to higher expected future productivity growth. Higher expected future productivity growth translates into higher real interest rates and lower prices for bonds. In a New Keynesian framework with recursive preferences, the sign of the correlation between stock returns and bond returns depend on the source of risk. Time-varying relative variance of the cyclical and trend shocks to productivity growth determines the conditional correlation between stock returns and bond returns.

Calibrations and simulations results support this mechanism. The changing magnitude and composition of cyclical and trend shocks perform well in explaining the conditional correlation between stock and bond returns. The model is calibrated to the volatility of cyclical and trend volatility of productivity growth over two samples of US data: pre-1998 and post-1998. I find that the volatility of cyclical productivity shock decreases by around 25 percent from the earlier period to the latter period while the volatility of trend productivity increases by around 30 percent. The calibrated model approximately matches the stock-bond correlation in both samples .

I employ a model that leaves out many of the nominal frictions in standard business cycle work in order to focus on the ability of the particular mechanism just described to generate realistic nominal and real stock-bond return correlation.

1.2 Some Descriptive Measures of Stock-Bond Return Comovement

This section summarizes some well-known, and some not so well-known, properties of stock and bond returns. Section 2.1 and Section 2.2 focus on U.S. and U.K. markets respectively.

1.2.1 U.S. Stock-Bond Return Correlation

Figure 1.1 displays yearly estimates of correlations between aggregate stock returns and returns to both nominal and inflation-indexed long-term Treasury bonds. Yearly estimates of correlation are produced using daily returns. Nominal returns are for the 10-year Treasury bond and real returns are for the 10-year Treasury inflation protected bonds (TIPS). The highest correlation between returns to stocks and returns to nominal bonds is 0.61 in year 1994, and the lowest correlation is -0.63 in year 2012. Guidolin and Timmermann (2006), Baele et al. (2010), Campbell et al. (2014) and other authors all highlight this striking pattern for nominal bonds shown in Figure 1.1 but don't examine correlation between returns to stocks and returns to real bonds. Returns to stocks and nominal bonds were positively correlated throughout the 1970s, 1980s, and the first half of the 1990s. Estimates of the correlation fluctuated over this period, but on average remain largely positive. In the latter half of the 1990s, estimated correlations dropped sharply to less than zero. Estimates have largely remained negative since then. The pattern carries over to returns calculated using longer holding periods. For example, Figure 1.2 displays estimates of correlations produced using monthly returns. The estimate for month t is the sample correlation of the 25 returns for months $t-12$ through $t+12$. Although details differ, the message in this figure matches that in Figure 1.1.

Researchers attempting to explain this large, persistent variation in the stock-nominal bond return correlation largely focus on the changing behavior of monetary policy and/or inflation over the sample. Campbell et al. (2014) argues that changing correlations are driven by regime shifts in the monetary policy reaction function. When the Fed tightens aggressively in response to unexpected increases of inflation, the stock-bond return correlation is more positive. In regimes when the Fed is more accommodating, the stock-bond correlation is more positive. Hasseltoft (2009) studies the implication of changing inflation volatility for stock-bond return correlation. Inflation is assumed to be negatively associated with consumption growth. David and Veronesi (2013) studies the joint dynamics of stock and bonds in an endowment economy with exogenous economic regimes, in which inflation could be either positively or negatively correlated with output growth.

However, evidence in Figures 1.1 and 1.2 casts considerable doubt on these stories. Returns to inflation indexed bonds are available beginning with their introduction by the Treasury in 1998. To my knowledge, this is the first paper that emphasizes the correlation between returns to stocks and returns to inflation indexed bonds. A striking result is that during this period estimated correlations of returns to stocks and returns to real bonds closely tracked the stock-nominal bond return correlations. The correlation between these two yearly series (i.e., the correlation between the two time series of yearly estimates of correlations) is 0.71. This tight link suggests that the fundamental determinants of time-varying correlation apply to both real and nominal bonds. It is of course possible that in the mid-1990s there was a large regime change associated with inflation, which cannot be detected using more recent data. We need a longer sample to examine this possibility.

1.2.2 U.K. Stock-Bond Return Correlation

In the United Kingdom, the history of real bonds goes back to 1986 when inflation was still relatively high. Figure 1.3 is the U.K. version of Figure 1.1, displaying yearly estimates of correlations between aggregate stock returns and returns to both nominal and real bonds using daily returns. Stock and nominal bond return correlations are examined by Gusset and Zimmermann (2015), but they do not extend their analysis to real bonds. Nominal returns are for the 10-year gilts and real returns are for the returns the 10-year inflation indexed gilts. [1] The highest correlation between returns to stocks and returns to nominal bonds is 0.59 in 1994, while the lowest correlation observed is -0.61 in 2011. Similar to the finding for U.S., returns to stocks and

[1] Gilts are bonds that are issued by the British government, which are UK equivalent of US Treasury securities. The data is available at http://www.bankofengland.co.uk/statistics/Pages/yieldcurve/archive.aspx

nominal bonds were largely positively correlated until the late 1990s, then largely negative. More importantly, the correlation between returns to nominal bonds and stocks is closely related to the correlation between returns to real bonds and stocks. The correlation between these two yearly series (i.e., the correlation between two time series of yearly estimates of correlations) is 0.97. The correlations between stock returns and inflation-indexed bond returns were also largely positive until the late 1990s, then turned negative. The pattern also applies to returns calculated using longer holding periods. For example, Figure 1.4 is the U.K. version of Figure 1.2. Estimates of correlations are produced from monthly returns. The message in this figure largely matches that in Figure 1.3, which are produced using daily returns. This tight link between returns of nominal and real bonds is consistent with Duffee (2016), which finds that variances of news about expected inflation account for between 10 to 20 percent of variances of yield shocks at a quarterly frequency.

1.3 The Model

How important are cyclical and trend fluctuations for macroeconomic quantities and prices? To answer this question, I develop a general equilibrium framework to quantitatively account for both macroeconomic and financial moments. It builds on the standard New Keynesian framework of Woodford (2003) and Galí (2009). There are three standard New Keynesian ingredients. First, the model features imperfect competition in the good market: each firm produces a differentiated good for which it sets the price, given a demand constraint. Second, Calvo (1983) type of price stickiness is introduced by assuming that only a fraction of firms can reset their prices in any given period. Third, the central bank in this economy sets the nominal interest rate according to a Taylor (1993) type rule.

Following the finance literature, households in the economy derive felicity from consumption and leisure following an Epstein and Zin (1989) and Weil (1989) type of utility function. By introducing Epstein-Zin preferences, the model separates the elasticity of intertemporal substitution and risk aversion coefficient and therefore better matches the asset pricing moments.

1.3.1 Firms

There exists a continuum of firms indexed by $i \in [0, 1]$. Each firm produces a differentiated good, but they all use an identical technology, represented by the production function

$$Y_t(i) = e^{z_t} K_t^{\alpha} (e^{\Gamma_t} N_t(i))^{1-\alpha} \tag{1.1}$$

where K_t is the capital stock. The aggregate "final" output is produced from individual goods such that

$$Y_t \equiv \left(\int_0^1 Y_t(i)^{1-\frac{1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} \tag{1.2}$$

where ϵ measures the degree of substitutability between individual goods.

The only shocks in this economy are shocks to productivity growth. The nature of these shocks drives the correlation between stock and bond returns. There are two general approaches in the literature to model shocks to productivity growth. One assumes that productivity growth follows a stationary process; thus the effects of shocks on productivity growth die out over time. This approach is seen in Rudebusch and Swanson (2012) and Kung (2015). Another assumes that productivity growth follows a unit root process, as seen in Croce (2014) and Hsu et al. (2016).

The production function (1.1) includes both kinds of shocks, which are common across firms. Their relative importance determines the sign of the stock-bond return correlation. The stationary process is z_t, with dynamics

$$z_t = \rho_z z_{t-1} + e^{\sigma_{z,t-1}} \epsilon_{z,t} \tag{1.3}$$

where $\epsilon_{z,t}$ represents independently and identically distributed draws from a normal distribution with zero mean and standard deviation of 1. Stationarity is imposed by $|\rho_z| < 1$.

The unit root process is Γ_t, with dynamics

$$\Gamma_{t+1} = \Gamma_t + g_t = \sum_{s=0}^{t} g_s$$

$$g_t = (1-\rho_g)\mu_g + \rho_g g_{t-1} + e^{\sigma_{g,t-1}}\epsilon_{g,t} \tag{1.4}$$

where $|\rho_g| < 1$, and $\epsilon_{g,t}$ represents independently and identically distributed draws from a normal distribution with zero mean and standard deviation 1. The term μ_g captures the long-run mean growth rate of technology.

The relative importance of the productivity shocks in (1.3) and (1.4) drives the sign of the stock-bond return correlation. The intuition is easiest to see through a comparative statics exercise by comparing two cases with fixed volatilities but where the relative importance of the cyclical and trend shock differ.

Fixed Volatility Assumption:

$$\sigma_{z,t} = \sigma_z, \quad \sigma_{g,t} = \sigma_g \tag{1.5}$$

The fixed volatility specification follows Aguiar and Gopinath (2007). My approach differs from theirs both in the focus (they examine capital flows of emerging markets) and in the choice of parameters.

The more general assumption is captured by [2]

[2] Naturally, the fixed volatility assumption is inconsistent with the motivating evidence that correlations change over time. It also oversimplifies the asset-pricing setting, since investors do not have to consider the possibility that relative volatilities will vary.

Stochastic Volatility Assumption:

$$\sigma_{z,t} = (1 - \rho_{\sigma z})\sigma_z + \rho_{\sigma z}\sigma_{z,t-1} + \eta_{\sigma,z}\epsilon_{\sigma z,t} \tag{1.6}$$

$$\sigma_{g,t} = (1 - \rho_{\sigma g})\sigma_g + \rho_{\sigma g}\sigma_{g,t-1} + \eta_{\sigma,g}\epsilon_{\sigma g,t} \tag{1.7}$$

The main feature of the process is that the log standard deviations $\sigma_{z,t}$ and $\sigma_{g,t}$ are not constants over time, as commonly assumed. The variation of $\sigma_{z,t}$ and $\sigma_{g,t}$ captures the stochastic volatility of cyclical and trend shocks respectively. The shocks $\epsilon_{\sigma z,t}$ and $\epsilon_{\sigma g,t}$ are normally distributed with mean zero and unit variance. The parameters σ_z (σ_g) and η_z (η_g) controls mean volatility and the standard deviation of shocks to volatility for the cyclical (trend) productivity volatility process. A high σ_z (σ_g) implies a high mean volatility of cyclical (trend) productivity process, and a high $\eta_{\sigma,z}$ ($\eta_{\sigma,g}$) implies large shocks to cyclical (trend) volatility. Croce (2014) studies a production economy with stochastic volatility where productivity growth follows a unit root. In Section 5, I consider this fully dynamic version of the model with stochastic volatility. Still, the fixed volatility model is sufficient to demonstrate the intuition.

1.3.2 Households

We assume that there exists a representative household with Epstein and Zin (1989) and Weil (1989) preferences over the consumption good C_t and leisure L_t with the utility function V_t satisfying:

$$V_t = \left\{(1-\beta)U(C_t, N_t) + \beta \mathbb{E}_t[V_{t+1}^{1-\gamma}]^{\frac{1-\psi}{1-\gamma}}\right\}^{\frac{1}{1-\psi}} \tag{1.8}$$

where γ is the risk aversion coefficient and ψ is the inverse of intertemporal elasticity of substitution. C_t is a consumption index given by

$$C_t \equiv \left(\int_0^1 C_t(i)^{1-\frac{1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}} \tag{1.9}$$

The instantaneous utility function is given by

$$U(C_t, N_t) = \begin{cases} \frac{C_t^{1-\psi}}{1-\psi} - e^{\chi(1-\psi)} e^{\Gamma_t(1-\psi)} \frac{N_t^{1+\varphi}}{1+\varphi} & \text{if} \quad \psi \neq 1 \\ \log(C_t) - e^{\chi(1-\psi)} e^{\Gamma_t(1-\psi)} \frac{N_t^{1+\varphi}}{1+\varphi} & \text{if} \quad \psi = 1 \end{cases} \tag{1.10}$$

where $\psi \geq 0$ and $\varphi \geq 0$ determine, respectively, the curvature of the utility of consumption and the disutility of labor. The analysis is considerably simplified by two properties of the above utility function: (1) separability, that is $U_{cn,t} = 0$ and (2) the implied constancy of the elasticities for the marginal utility of consumption and for the marginal disutility of labor. The term $g_t^{1-\psi}$ is introduced to make the utility function consistent with the notion of balanced growth path as seen in Rudebusch and Swanson (2012). The parameter N_t denotes hours of work or employment. Parameter $\beta \in (0, 1)$ is the discount factor. The notation $\mathbb{E}_t\{.\}$ denotes the expectational operator, conditional on information at time t.

The key advantage of using Epstein-Zin utility is that it breaks the link between intertemporal elasticity of substitution and the coefficient of relative risk aversion that has long been noted in the literature regarding expected utility see, e.g., Weil (1989). Household risk aversion to uncertain lotteries over V_{t+1} is amplified by the additional parameter γ, a feature which is crucial for allowing us to fit both the asset pricing and macroeconomic facts below. Note, when $\gamma = \psi$, the utility function coincides with the usual CRRA utility function.

1.3.2.1 The Marginal Rate of Substitution

The marginal rate of substitution (MRS) between neighboring dates in this economy is given by (see, e.g. Hansen et al. (2007)), [3]

$$M_{t,t+1} = \beta \frac{U_{C,t+1}}{U_{C,t}} \left[\frac{V_{t+1}}{(\mathbb{E}_t V_{t+1}^{1-\gamma})^{1/1-\gamma}} \right]^{\psi-\gamma} \tag{1.11}$$

In the case of $\gamma = \psi$, $M_{t,t+1}$ reduces to the usual formula for the marginal rate of substitution when utility depends only on current period consumption. Therefore, my preference specification nests the class of preferences studied by King et al. (1988).

It is useful to consider an asset that pays C_t as its dividend in each period. This asset is a claim to all future consumption streams $C_{t+1}, C_{t+2}, ...$ In the usual analysis of Epstein-Zin preferences, one substitutes the return on an asset that pays

[3] Detailed derivation is provided in appendix as well.

consumption as its dividend into the MRS. Denote the ex-dividend price of this asset as $W_{U,t}$. The return for this asset from t to $t+1$ is defined as

$$R_{W,t+1} = \frac{C_{t+1} + W_{U,t+1}}{W_{U,t}} \tag{1.12}$$

The appendix shows that the stochastic discount factor (1.11) can be expressed using the return on this asset as

$$M_{t,t+1} = \left(\beta \frac{U_{C,t+1}}{U_{C,t}}\right)^{1-\chi} \left(R_{w,t+1}^{-1}\right)^{\chi} \tag{1.13}$$

The logarithm of the marginal rate of substitution (MRS) is

$$m_{t+1} = (1-\chi)\rho - (1-\chi)\psi\Delta c_{t+1} - \chi r_{c,t+1}$$

where $1 - \chi = \frac{1-\gamma}{1-\psi}$.

The expression for the marginal rate of substitution in terms of an asset return is useful for two reasons. First, expressing the marginal rate of substitution in terms of asset returns will be important in the implementation of the approximation method for the model. Second, it shows how the marginal rate of substitution changes from the usual form by introducing Epstein-Zin preferences. Instead of the standard setup where only consumption matters, the marginal rate of substitution now depends on the realization of the asset returns.

1.3.2.2 Budget Constraint

The maximization of utility (1.8) is subject to a sequence of flow budget constraints given by

$$P_t \left[C_t + K_{t+1} - (1-\delta)K_t + \frac{\phi}{2}\left(\frac{K_{t+1}}{K_t} - e^{\mu_g}\right)^2 K_t \right] + q_t B_t = W_t N_t + D_t + B_{t-1} \tag{1.14}$$

Capital depreciates at the rate δ, and changes to the capital stock entail a quadratic adjustment cost [4]

$$\frac{\phi}{2}\left(\frac{K_{t+1}}{K_t} - e^{\mu_g}\right)^2 K_t$$

in which $t = 0, 1, 2, ...$ The parameter P_t is the price of the consumption good, and W_t denotes the nominal wage (per hour or per worker, depending on the interpretation of N_t). The symbol B_t represents the quantity of one-period nominally riskless discount bonds purchased in period t and maturing in period $t+1$. Each bond pays one unit of money at maturity, and its price is Q_t. Nominal dividends are represented by D_t, accruing to households as the owner of firms. In addition to (1.14), it is assumed that households are subject to solvency constraint that prevent them from engaging

[4] This form of adjustment cost is motivated to ensure there is no adjustment costs in the steady state.

in Ponzi-type schemes. The following constraint is assumed

$$\lim_{T\to\infty} \mathbb{E}_t \left\{ M_{t,T} \frac{B_T}{P_T} \right\} \geq 0 \tag{1.15}$$

for all t, where $M_{t,T} \equiv \beta^{T-t} U_{c,T}/U_{c,t}$ is the stochastic discount factor. We also use S_t to denote the market value of firms' shares.

1.3.3 Optimal Price Setting

Following the formalism proposed in Calvo (1983), each firm may reset its price only with probability $1-\theta$ in any given period, independent of the time elapsed since it last adjusted its price. Thus, in each period a measure of $1-\theta$ producers reset their prices, while a fraction of θ keep them unchanged. As a result, the average duration of a price is given by $\frac{1}{1-\theta}$. Therefore, θ is the measure of price stickiness.

A firm reoptimizing in period t, will choose P_t^* that maximizes the current market value of the profits generated while that price remains effective.

$$\max_{P_t^*} \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left\{ M_{t,t+k}^{\$} (P_t^* Y_{t+k|t} - \Psi_{t+k}(Y_{t+k|t}) \right\} \tag{1.16}$$

subject to the sequence of demand constraints

$$Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}} \right)^{-\epsilon} Y_{t+k} \tag{1.17}$$

for $k = 0, 1, 2, ...,$, where $M^{\$}_{t,t+k} \equiv \beta^k (U_{c,t+k}/U_{c,t})(P_t/P_{t+k})$ denotes the nominal stochastic discount factor, and $\Psi_t(.)$ is the cost (nominal) function and $Y_{t+k|t}$ denotes output in period $t+k$ for a firm that last reset its price in period t.

1.3.4 Central Bank

The central bank in the economy sets the nominal interest rate following a Taylor (1993) policy rule

$$i_t = \rho_i i_{t-1} + (1 - \rho_i)[r^* + \phi_y \tilde{y}_t + \phi_\pi \pi_t] \tag{1.18}$$

where ϵ_v is an independently and identically distributed stochastic monetary policy shock with mean zero and variance σ_v^2, $\tilde{y}_t$ denote the deviation of detrended output from its steady state value where $y_t \equiv \log(Y_t) - \Gamma_t$. Coefficients ϕ_π and ϕ_y are chosen by the monetary authority, and assumed to be non-negative. r^* is the steady state level of real interest rate.

1.3.5 Equilibrium

In equilibrium, nominal wage W_t, the price of goods $P_{i,t}$ and consumption sector inflation π_t are set to clear all markets

- Labor market clearing:

- Consumption-good market clearing:

$$C_t + I_t = Y_t \tag{1.19}$$

- Zero net supply of bonds:

$$i_t = -\mathbb{E}_t[m^{\$}_{t,t+1}] \tag{1.20}$$

An equilibrium consists of prices and allocations such that (a) taking prices and wage as given, each household's allocations solves (1.8); (b) taking aggregate prices and wage as given, firm's allocation solve (1.16) ; (c) labor, consumption-good and bond markets clear. I'm solving for a symmetric equilibrium, in which all intermediate good firms choose the same price P_t, employ the same amount of labor N_t and choose to hold the same amount of capital K_t.

1.3.6 Equity Pricing

In models similar to this, there is always a question about what in the model should be taken as a proxy for the real world return to equities. I'll use a standard approach from the asset pricing literature that the stock market in this model is a leveraged claim on future aggregate consumption. In each period, it pays out consumption units D_t. The log of the aggregate dividend is scaled log consumption.

$$d_t = \phi c_t \tag{1.21}$$

The parameter ϕ is capturing a broad concept of leverage, including operating leverage. The interpretation of dividends as a levered claim on consumption is common in the asset pricing literature (Abel (1990), Campbell (2003), Bansal and Yaron (2004) and etc).

Let $W_{S,t}$ be the real price of stocks, the law of one prices implies that

$$W_{S,t} = \sum_{s=1}^{\infty} \mathbb{E}_t[M_{t,t+s} D_{t+s}] = \mathbb{E}_t[M_{t,t+1}(D_{t+1} + W_{S,t+1})] \tag{1.22}$$

1.3.7 Bond Pricing

The Euler equation implies that the price of nominal bonds satisfies that

$$\mathcal{P}^{\$}_{n,t} = \mathbb{E}_t(M_{t,t+1} e^{-\pi_{t+1}} \mathcal{P}^{\$}_{n-1,t+1}) \tag{1.23}$$

where $\mathcal{P}^{\$}_{n,t}$ is the price of a zero-coupon bond that matures on date $t+n$ and pays 1 dollar at time t.

The yield-to-maturity on the n -period nominal bond is defined as

$$\mathcal{Y}_{n,t}^{\$} = -\frac{1}{n}\mathcal{P}_{n,t}^{\$} \tag{1.24}$$

Similarly, the price of a n -period real bond can be written as

$$\mathcal{P}_{n,t} = \mathbb{E}_t[M_{t,t+1}\mathcal{P}_{n-1,t+1}] \tag{1.25}$$

and the corresponding yield-to-maturity is defined as

$$\mathcal{Y}_{n,t} = -\frac{1}{n}\mathcal{P}_{n,t} \tag{1.26}$$

1.4 Quantitative Implications

This section discusses the quantitative implications of the model. The model analyzed in this section is the one without stochastic volatility. The model is solved in Dynare using a first-order approximation. I find that solving the model with higher order approximation doesn't change much for the stock-bond correlation and other macro moments I focus on. There are two reasons for this result. First, the risk premium is small and vary little in the model. Second, risk premium and nonlinearity play little role in determining the key macro and financial moments here.

1.4.1 Data and Summary Statistics

I use quarterly US data on output, inflation, interest rates, and aggregate stock returns from 1960Q1-2015Q4. The productivity measure used is the labor productivity measure from the Bureau of Labor Statistics. [5]

1.4.2 Calibration

Table 1.1 presents the quarterly calibration for parameters of the model. Panel A reports the values for the preference parameters. The elasticity of intertemporal substitution σ is set to 2.0 and the coefficient of relative risk aversion is set to 10.0, both of which are standard values in the long-run risk literature (e.g., Bansal and Yaron (2004)). The subjective discount factor is calibrated to 0.99 to be consistent with the level of the real (risk-free) short-term rate.

Panel B reports the calibration of the technological parameters. The desired markup is set to be 1.2 (corresponding to an average markup of 1.2.). The capital share α is set to 0.33, and the depreciation rate of capital is set to 0.02. These three parameters are calibrated to standard values in the macroeconomic literature. The price adjustment parameter θ is set to be 0.75 meaning that 25 percent of firms adjust their prices in each period.

Panel C reports the parameter values for the productivity process. The persistence parameter ρ_z is calibrated to 0.95 to match the first autocorrelation of expected

[5]The data is available at http://download.bls.gov/pub/time.series/pr/

productivity growth. This value is in line with Rudebusch and Swanson (2012) and Kung and Schmid (2010)

Panel D reports the calibration of the monetary policy rule parameters. The parameter governing the sensitivity of the interest rate to inflation ρ_π is set to 1.5. The parameter determining the sensitivity of the interest rate to output ρ_y is set to 0.10. The volatility of interest rate shocks σ_v is set to 0.3%. Ang et al. (2009a) use a no-arbitrage term-structure model to estimate the monetary policy loadings on output gap and inflation. They find that the monetary policy loading on the output gap has averaged around 0.4 and has not changed very much over time. The inflation loading has changed substantially over the last 50 years and ranges from close to zero in 2003 to a high of 2.4 in 1983. My calibration is consistent with parameter estimates from the literature.

1.4.3 Evaluating the Fit of the Model

The goal of the current exercise is to see in a comparative static sense whether the model reproduces observed stock-bond return correlation. The model is calibrated to two periods of productivity growth differing only in the volatility of cyclical and trend shocks. I find that it can both provide a reasonable fit to the usual business cycle properties of the data, and importantly produce the striking change in the stock-bond return correlation discussed above. Table 1.2 summarizes the model fit for two subperiods of US economy: pre-1998 and post-1998.

1.4.3.1 Estimating the Volatility of the Cyclical and Trend Shock

The magnitude of cyclical and trend shock volatility for both samples is estimated from productivity data in pre-1998 and post-1998. In each sample, I conduct a maximum-likelihood estimation of the productivity processes for the volatility of the cyclical and trend shock. The estimates of the volatility are directly fed into the model. They are reported in the first two rows in Table 1.2. The volatility of the cyclical shock is about 12 times larger than the volatility of the trend shock in the pre-1998 sample. In the post-1998 sample, the volatility of cyclical shock is about 7 times larger than that of the trend shock.

1.4.3.2 Evaluating the General Adequacy to Macroeconomics Moments

Panel A shows that the model fits standard deviations of detrended output growth, inflation, and detrended wage rates moderately well. [6] For example, the standard deviation of detrended output is about 1.62 percent in the pre-1998 sample whereas the model produces 0.8 percent. Panel B shows the model could also moderately match the first-order autocorrelations of detrended output and inflation.

[6] I use a Hodrick and Prescott (1997) filter with smoothing parameter 1600 to detrend output and wages.

1.4.3.3 Evaluating the Adequacy to Key Financial Moments

Panel C shows the model approximately matches the key financial moment: stock-bond return correlation. In the pre-1998 sample, the model produces a stock-bond correlation of 0.27 while the correlation the data is 0.37. In the post-1998 period, the model produces a stock-bond correlation of -0.20 while it is -0.27 in the data. Another important feature of data that the model is able to approximately fit is the correlation between changes in yields and changes in the slope of the yield curve. [7] Because cyclical shocks are mean-reverting and short-lived, they have larger effects on short-term interest rates relative to long-term ones. On the contrary, trend shocks are long-lived and therefore have bigger effects on the long-term interest rates. Therefore, the slope of the yield curve decreases in response to a positive cyclical shock whereas it increases in response to a positive trend shock. The correlation between changes in long-term interest rates and changes in the slope of the yield curve is thus negative following cyclical shocks and positive with respect to trend shocks.

Figure 1.5 plots the 5-year moving correlation between changes of the 5-year zero coupon bond yields and the slope of the yield curve. [8] The correlation is mostly negative in the 1970s and 1980s and becomes positive in the recent decades. This pattern is striking as it is the opposite of the movement we see for stock-bond returns correlation. Nevertheless, this pattern is not implied by the correlation between

[7] The slope of the yield curve, in general, is defined as the long-term interest rates minus the short-term interest rates.

[8] The slope is measured by the 5-year yield less the three-month bill rate.

stock-bond returns. Therefore, it could serve as an important validation for the key mechanism of the model. I evaluate the fit of the model to the correlation between changes in 5-year yields and changes in yield curve slopes. In the pre-1998 period, the model produces a correlation of -0.08 compared with -0.22 in the data. In the post-1998 period, the model produces a correlation of 0.28 compared with 0.49 in the data. The model underpredicts the correlation in the post sample. A potential reason is that the short-term interest rate in the U.S. has been stuck at the zero-bound in recent years whereas the model has no zero lower bound. Therefore, changes of slopes are strongly positively correlated with changes of long-term yields.

1.4.4 Impulse Responses

Impulse response functions clarify the mechanism by which individual shocks act on stocks, bonds, and macroeconomic variables. This section presents the impulses of variables to three exogenous shocks.

Figure 1.6 shows responses of output, inflation, nominal interest rate, the yield for 10-year nominal and real bonds and stock prices to a cyclical technology shock. A technology shock acts as a strongly positive impulse to output, but as a negative one to output gap and inflation. It increases output, lowers the marginal production cost and therefore inflation. Long-term real interests fall as people expect the economy to go back to the long-term trend. Both stock and bond prices increase following a cyclical technology shock, so technology shocks tend to raise stock-bond correlation.

Figure 1.7 shows responses of output, inflation, nominal interest rate, the yield for 10-year nominal and real bonds and stock prices to a a trend shock. A trend shock acts as a strongly positive impulse to nominal and real short-term interest rates. Because it implies a large wealth effect to consumers, households increase consumption. The output gap and inflation, therefore, increase as well. When the intertemporal elasticity of substitution is low meaning that people are willing to substitute consumption over time, stock prices increases significantly following a trend shock. Therefore, stock prices increase while bond prices decrease following a trend shock. Thus, trend shocks tend to decrease stock-bond return correlation.

1.5 Investigating the Mechanisms of the Model with Stochastic Volatility

This section investigates the mechanism of the model with stochastic volatility, which is described in Assumption 2. Although I solve the model numerically, I demonstrate the mechanisms working in my model via approximate analytically solutions. Let the vector θ collect all parameters of the model (other than the standard deviations of the structural disturbances of the time-invariant model). I log-linearize the Euler equations that are directly associated with macro quantities. [9] The solution of

[9] It has been shown in the literature that for instance, Tallarini (2000), risk aversion coefficient does not significantly affect the relative variability and comovements of aggregate quantity variables. Therefore, in solving the macro quantities, I take the first-order approximation of the model around the non-stochastic steady state. All the macro quantities don't depend on the risk-aversion

the model leads to a state-space representation of the form

$$x_t = \mu_x + D(\theta)s_t$$

$$s_t = G(\theta)s_{t-1} + H(\theta)\omega_t$$

where x_t represents a vector of observable variables and s_t denotes the vector of endogenous/state variables in log-deviation from the deterministic steady state. Note, I don't include state variables that only describe volatility dynamics in s_t.

Indexing each structural shock by i, the stochastic volatility for each shocks is expressed as

$$\omega_{i,t} = \sigma_{i,t-1}\epsilon_{i,t}$$

$$\sigma^2_{i,t} = (1-\rho_{\sigma,i})\sigma_i + \rho_{\sigma,i}\sigma^2_{i,t-1} + \eta_{\sigma,i}\epsilon_{\sigma i,t}$$

To put the dynamics of volatility in matrix forms, define $\sigma_t = [\sigma_{i,t}]'$, $\sigma_0 = [\sigma_i]'$, $T_\sigma = diag[\rho_{\sigma,i}]$, $\eta = diag([\eta_{\sigma,i}])$, and $\epsilon_\sigma = [\epsilon_{\sigma,i}]'$ we can write

$$\sigma^2_{t+1} = \sigma^2_0 + T_\sigma(\sigma^2_t - \sigma^2_0) + \eta\epsilon_\sigma \tag{1.27}$$

coefficient.

I use the standard approximations utilized in Campbell and Shiller (1988),

$$r_{c,t+1} = \kappa_0 + \kappa_1 \mathcal{Q}_{t+1} - \mathcal{Q}_t + \Delta c_{t+1} \tag{1.28}$$

where κ_0 and κ_1 are approximating constants that both depend only on the average level of q. Analogously, $r_{m,t+1}$ and $q_{m,t+1}$ correspond to the market return and its log price-dividend ratio.

The logarithm of the intertemporal marginal rate of substitution (IMRS) is

$$m_{t+1} = (1-\chi)\rho - (1-\chi)\psi\Delta c_{t+1} - \chi r_{c,t+1}$$

where $1-\chi = \frac{1-\gamma}{1-\psi}$. It follows that the innovation in m_{t+1} is driven by the innovations in Δc_{t+1} and $r_{c,t+1}$. Covariation with innovation in m_{t+1} determines the risk premium for any asset. When the inverse of elasticity of risk aversion equals the risk aversion coefficient, $\chi = 0$ and the IMRS collapses to the usual case of power utility.

By the benefit of the linear structure for macro quantities, the consumption growth Δc_{t+1} can expressed as

$$\Delta c_{t+1} = \mu_c + T_c' s_t + \zeta_c' \omega_{t+1} \tag{1.29}$$

Similarly, the dividend growth for the stock, which is a leveraged claim on consumption growth follows

$$\Delta d_{t+1} = \mu_d + T_d' s_t + \zeta_d' \omega_{t+1} \tag{1.30}$$

where $\mu_d = \phi\mu_c$, $T_d = \phi T_c$ and $\zeta_d = \phi\zeta_c$.

We assume that the price-consumption ratio can be expressed as

$$\mathcal{Q}_t = A_0 + A_1' s_t + A_2' \sigma_t^2$$

This form for $\mathcal{Q}_t$ is obtained by exploiting the Euler equation. An analogous expression holds for the log price-dividend ratio $\mathcal{Q}_{m,t}$ where

$$\mathcal{Q}_{m,t} = A_{0,m} + A_{1,m}' s_t + A_{2,m}' \sigma_t^2$$

We substitute the form of price-consumption ratio, consumption/dividend growth into the Euler equation, matching coefficients on terms on state variables, we can obtain the above formula for stock prices: [10]

Using the expressions for the real and nominal discount factors, we can solve for equilibrium yields in the economy. the real and nominal yields are linear in the

[10] The details of derivations are provided in the Appendix.

economic state variables:

$$\mathcal{Y}_{t,n} = \frac{1}{n}(B_{0,n} + B'_{1,n}s_t + B'_{2,n}\sigma_t^2) \tag{1.31}$$

$$\mathcal{Y}^{\$}_{t,n} = \frac{1}{n}(B^{\$}_{0,n} + B^{\$\,\prime}_{1,n}s_t + B^{\$\,\prime}_{2,n}\sigma_t^2) \tag{1.32}$$

The expressions are also obtained from the Euler equation and our dynamics for IMRS.

1.5.1 Stock-Bond Return Correlation

The (nominal) return for holding a n-period real bond for one period from t to $t+1$ is

$$\begin{aligned} r_{n,t+1} &= -(n-1)\mathcal{Y}_{t+1,n-1} + n\mathcal{Y}_{t,n} + \pi_{t+1} \\ &= B_{0,n} + B'_{1,n}s_t + B'_{2,n}\sigma_t^2 - B_{0,n-1} - B'_{1,n-1}s_{t+1} - B'_{2,n-1}\sigma_{t+1}^2 + \pi_{t+1} \end{aligned}$$

and for nominal bonds

$$\begin{aligned} r^{\$}_{n,t+1} &= -(n-1)\mathcal{Y}^{\$}_{t+1,n-1} + n\mathcal{Y}^{\$}_{t,n} \\ &= B^{\$}_{0,n} + B^{\$\,\prime}_{1,n}s_t + B^{\$\,\prime}_{2,n}\sigma_t^2 - B^{\$}_{0,n-1} - B^{\$\,\prime}_{1,n-1}s_{t+1} - B^{\$\,\prime}_{2,n-1}\sigma_{t+1}^2 \end{aligned}$$

The covariance of the (nominal) return to the aggregate consumption claim and the return to the real bond is

$$Cov_t(r_{c,t+1} + \pi_{t+1}, r_{n,t+1}) = -Cov_t(r_{c,t+1}, \mathcal{Y}_{t+1,n-1}) + Var_t(\pi_{t+1})$$
$$= -B'_{1,n-1}\sigma_t\sigma'_t(\kappa_1 A'_1 H + \zeta'_c)' - \kappa_1 A_2 \eta\eta' B_{2,n-1} + Var_t(\pi_{t+1})$$

Inflation is also linear in state variables

$$\pi_t = \mu_\pi + T'_\pi s_t + \zeta_\pi \omega_{t+1} \tag{1.33}$$

Similarly, the covariance of the nominal return to aggregate consumption claim and the return to the nominal bond is

$$Cov_t(r_{c,t+1} + \pi_{t+1}, r^{\$}_{n,t+1}) = -Cov_t(r_{c,t+1}, (n-1)\mathcal{Y}^{\$}_{t+1,n-1}) - Cov_t(\pi_{t+1}, (n-1)\mathcal{Y}^{\$}_{t+1,n-1})$$
$$= -B^{\$\,\prime}_{1,n-1} H' \sigma_t\sigma'_t(\kappa_1 A'_1 H + \zeta'_c)' - \kappa_1 A_2 \eta\eta' B^{\$\,\prime}_{2,n-1}$$

The covariance of the (nominal) return to the stock and the return to the real bond is

$$Cov_t(r_{m,t+1} + \pi_{t+1}, r_{n,t+1}) = -Cov_t(r_{m,t+1}, \mathcal{Y}_{t+1,n-1}) + Var_t(\pi_{t+1})$$
$$= -B'_{1,n-1} H' \sigma_t\sigma'_t(\kappa_{1,m} A'_{1,m} H + \zeta'_d + H T_\pi)' - \kappa_{1,m} A_{2,m} \eta\eta' B_{2,n-1} + Var_t(\pi_{t+1})$$

The covariance of the nominal return to the stock and the return to the nominal bond is

$$Cov_t(r_{m,t+1} + \pi_{t+1}, r^{\$}_{n,t+1}) = -Cov_t(r_{m,t+1} + \pi_{t+1}, (n-1)\mathcal{Y}^{\$}_{t+1,n-1})$$

$$= -B^{\$\prime}_{1,n-1} H' \sigma_t \sigma_t' (\kappa_{1,m} A'_{1,m} H + \zeta'_d + H\zeta_\pi)' - \kappa_{1,m} A_{2,m} \eta \eta' B^{\$\prime}_{2,n-1}$$

The main insight from the analytical solutions is there are two sources determining the stock-bond correlation: the contemporaneous level of volatility of both cyclical and trend components and shocks to cyclical and trend volatility. In my quantitative exercise, I find that shocks to cyclical and trend volatility are small but very persistent. The stock-bond correlation is largely determined by the contemporaneous level of cyclical and trend volatility, seen the $\sigma_t \sigma_t'$ term in the previous equation.

1.5.2 Term Premium

Since the conditional mean of the log stochastic discount factor is linear in economic states, the innovation in the stochastic discount factor that determines the sources and the compensation for risk in the economy is given by

$$m_{t,t+1} - \mathbb{E}_t m_{t,t+1} = -\lambda_s \omega_{t,t+1} - \lambda_\sigma \epsilon_\sigma \tag{1.34}$$

In the model, the one-period expected excess return on nominal bonds can be written in the following following form:

$$\mathbb{E}_t(rx^{\$}_{t\to t+1,n}) + \frac{1}{2}Var_t(rx^{\$}_{t\to t+1,n}) = -Cov_t(m^{\$}_{t,t+1}, rx^{\$}_{t\to t+1,n}) \tag{1.35}$$

$$= const - B^{\$}_{1,n-1}\sigma_s\sigma_s'\lambda_s' - B^{\$}_{2,n-1}\eta_\sigma\eta_\sigma'\lambda_\sigma' \tag{1.36}$$

1.6 Estimation

In a dynamic setting, measuring cyclical and trend components from productivity and macroeconomic series is not an easy task in small samples. However, the ability to estimate such components of productivity at each point in time plays a crucial role in the testing of my model against the data. In this section, I develop a method to quantify the variations of the cyclical and trend component using both productivity data and asset prices data.

I focus the attention on quarterly series because the identification of highly persistent requires time series as long as possible.

Due to the nonlinearity embedded in the stochastic volatility setup of the model, Kalman filter is not useful here. First, the presence of stochastic volatility leads to fat tail on the distribution of observed variables, which renders the use of Kalman filter unsuitable. Second, the law of motion for the underlying states of the economy is inherently non-linear. I estimate the model using particle filters, which have been proposed by Fernndez-Villaverde and Rubio-Ramrez (2007), Herbst and Schorfheide (2015) and other authors as a good method to handle models with nonlinearity. [11] The prior distribution for the volatility parameters are summarized in the table. Due to the computation complexity involved the model, it is computationally burdensome to jointly estimate all model parameters. Hence, the exogenous stochastic driving processes of the model featuring time-varying volatility are first estimated using the particle filter method. The remaining parameters are calibrated using standard values from the literature.

1.6.1 Productivity-Only Information

Historically, U.S labor productivity growth (defined as output per hour worked) in the business sector has varied greatly. The strong growth rate of 3.3% in the period of 1947-1973 was followed by a sharp slowdown to 1.6% in the two decades that followed. The information and communication technology (ICT) boom in period 1996−2003 led to the "productivity miracle", when labor productivity growth doubled. As the gains from the ICT boom had largely been reaped, productivity growth slowed down to 1.9% in the pre-crisis years (2004-2007). Labor productivity growth has been disappointing since the crisis. I estimate the stochastic volatility processes for productivity in the model.

[11] The details of the particle filtering is contained in the appendix. Fernndez-Villaverde and Rubio-Ramrez (2007) estimate a business cycle model with investment-specific technological change, preference shocks, and stochastic volatility. Creal and Wu (2014) estimate a term structure model with two sources of uncertainty.

The dynamics of the state space is specified in the model section. I specify that the persistence of cyclical component is z_t is 0.95 following Rudebusch and Swanson (2012) and the persistence of of the growth rate g_t is 0.95 following Bansal and Yaron (2004). The reason of specifying these two parameters is the productivity data alone cannot provide enough information to separate the cyclical and trend component without knowing their persistence. The appendix contains detailed explanations of particle filtering.

The following graph 1.8 displays the result. There is a downward trend for cyclical volatility in the sample. The highest cyclical volatility in the 80s about 2.7 times larger than the lowest cyclical volatility in early 2000s. The trend volatility has seen an upward trend over the sample, even though the confidence interval is wider. The imprecision is mainly limited by the ability to pin down trend component volatility from the productivity series alone. In the next section, I further incorporate the realized stock-bond correlation to extract information about volatility from asset prices.

1.6.2 Incorporate Realized Stock-Bond Return Correlation Information

In this section, I estimate parameters associated with volatility processes of the model using two observables: productivity growth and realized stock-bond return correlation. The data is from 1971:Q1 to 2015:Q4. My sample length is limited by the availability of high frequency (daily) data on long-term bond yields. The high frequency data is used to construct estimates of stock-bond correlation. In this section, I perform the estimation exercise from both a classical perspective and from a Bayesian one. For the classical perspective, I maximize the likelihood of the model with respect to the parameters. For the Bayesian approach, I specify prior distributions over parameters, evaluate the likelihood using particle filter, and draw from the posterior distribution using a Metropolis-Hastings algorithm. The results from both approaches are very similar.

1.6.2.1 Classical Approach

Before the estimation, I constrain some parameter values. First, I constrain the parameters $\{\eta_{\sigma z}, \eta_{\sigma,g}\}$ determining S.D. must be positive. Second, the autoregressive coefficients $\{\rho_{\sigma z}, \rho_{\sigma,g}\}$ will be between 0 and 1 to maintain stationarity.

Table 1.3 reports the maximum likelihood estimation of the six parameters associated with the volatility process for the model. The autoregressive coefficients of cyclical component volatility and trend component volatility $\{\rho_{\sigma z}, \rho_{\sigma,g}\}$ reveal high persistence of volatility fluctuations. The autoregressive coefficients estimates are consistent with information of stock-bond return correlation. The sample standard deviation of stock-bond return correlation is 0.39. I also construct the standard deviation of stock-bond return correlation from simulated samples. The samples are of the same length as my data. The standard deviation from the simulated sample is in the close neighborhood of 0.39. Therefore, the high persistence of volatility fluctuation is not inconsistent with data.

The standard deviation of shocks to volatility $\{\eta_{\sigma z}, \eta_{\sigma,g}\}$ are of the same scale, even though the level of volatility of trend component is smaller than that of the cyclical component. The estimates of $\{\eta_{\sigma z}, \eta_{\sigma g}, \sigma_z, \sigma_g\}$ are relatively not precisely estimated. With quarterly data, it is difficult to pin down these volatility parameters.

1.6.2.2 Bayesian Approach

Table 1.4 reports the estimates from a Bayesian perspective. It lists the estimated parameters and their priors in the estimation. For the volatility processes parameters, I use relatively flat priors. For the persistence of volatility process, I choose a beta distribution with a mean of 0.9 and standard deviation of 0.1. The prior distribution for the standard deviation of shocks to volatility processes is gamma distributed with mean 0.2 and standard deviation of 0.1. I specify a uniform prior distribution for the mean of volatility process. The estimation result from the Bayesian approach is similar to the maximum likelihood one. It points to very persistent processes for both cyclical and trend volatility, with persistence of 0.999 and 0.998. The standard deviations for posterior distributions of $\{\sigma_z, \sigma_g\}$ is close to those of my prior distributions, suggesting that it is hard to pin down volatility parameters by only using quarterly data.

1.6.3 Evolution of Volatility

In Figure 1.9, I plot the mean of the smoothed paths of the volatility of the cyclical component and trend component. In the graphs, I also plot the 1 S.D. bands around the mean of the smoothed paths. The bands illustrate that my smoothed path is not imprecisely estimated.

The top figure in Figure 1.9 shows that the evolution of the volatility of the cyclical shock. The volatility of this shock has significantly declined over the sample. It is high in the 1970s, 1980s and early 1990s. The bottom figure in Figure 1.9 indicates the volatility of the trend shock has increased from late 1990s. My model attributes the changing sign of stock-bond return correlation to the product of both a decline in the cyclical shock volatility and an increase of the trend shock volatility.

1.7 Conclusion

In this paper, I show that time-varying relative volatility of cyclical and trend shocks to productivity can account for changes in observed stock bond correlation. The cyclical fluctuations of productivity growth led to large positive comovement between stock and bond returns, while the trend fluctuations lead to large negative comovement between stock and bond returns. I find from the U.S productivity data that the volatility of cyclical productivity shock decreases by around 25% and the volatility of trend productivity increases by around 30% from the pre-1998 sample to the post-1998 sample.

I propose a New Keynesian model featuring changing volatility of cyclical and trend shocks. By providing a macroeconomic link between output dynamics, interest rates, and inflation, the model may be helpful for investors and economists to better understand the determinants of Treasury bond risks, and for policy makers in the execution of monetary policy. This paper argues that the heteroscedasticity of cyclical and trend shocks are important macroeconomic drivers of stock-bond return correlation. The calibrated model is capable of replicating the variations of the stock-bond returns correlation.

The main mechanism identified by the model provides several additional testable implications, such as the covariance between short and long term interest rates. The cyclical fluctuations are associated with more movement in short-term interest rates relative to that in the long-term rates. The trend fluctuations are associated with more movements in long-term interest rates. This pattern is broadly consistent with the data.

1.8 Tables and Figures

Table 1.1: Quarterly Calibration

Parameter	Description	Value
Panel A: Preferences		
β	Subjective discount factor	0.99
σ	Inverse of elasticity of intertemporal substitution	0.5
CRRA	Risk aversion	10.0
φ	Inverse of labor supply elasticity	0.3
Panel B: Technology		
α	Capital share	0.33
δ	Depreciation rate of capital stock	0.02
θ	Price adjustment frequencies	0.75
Panel C: Productivities		
ρ_z	Persistence of z	0.90
ρ_g	Persistence of g	0.85
Panel D: Monetary Policy		
ρ_i	Degree of monetary policy inertia	0.7
ϕ_y	Sensitivity of interest rate to output	0.1
ϕ_π	Sensitivity of interest rate of inflation	1.5

This table reports the parameter values used in the quarterly calibration of the model. The table is divided into four categories: preferences, technology, firms price setting and policy parameters.

Table 1.2: Calibration

Sample Period	1960Q1-1998Q4		1999Q1- 2015Q4	
Statistic	Data	Model	Data	Model
Volatility Parameters				
σ_z	0.81	0.81	0.60	0.60
σ_g	0.068	0.068	0.091	0.091
Panel A: Standard deviations				
$\sigma(y)$	1.62	0.8	1.20	0.65
$\sigma(\pi)$	2.50	2.32	0.94	1.45
$\sigma(w)$	1.11	0.46	1.56	0.80
Panel B: Autocorrelations				
$AC_1(y)$	0.98	0.98	0.94	0.98
$AC_1(\pi)$	0.88	0.95	0.50	0.96
Panel C: Correlations				
$corr(\Delta\mathcal{Y}_5^\$ - \Delta\mathcal{Y}_{3m}^\$, \Delta\mathcal{Y}_5^\$)$	-0.22	-0.08	0.49	0.28
$corr(\pi, \Delta c)$	-0.32	-0.25	0.11	0.01
$corr(r_w, r_{10}^\$)$	0.37	0.27	-0.25	-0.20

This table presents the standard deviations, autocorrelations, and cross-correlations for key economic variables from the data and from the model. The model is calibrated at a quarterly frequency and the reported statistics are annualized.

Table 1.3: Maximum Likelihood Estimates

Parameter	Point estimate
Cyclical Shock	
$\rho_{\sigma z}$	0.999
σ_z	-4.6
$\eta_{\sigma z}$	0.08
Trend Shock	
$\rho_{\sigma g}$	0.998
σ_g	-6.5
$\eta_{\sigma g}$	0.10

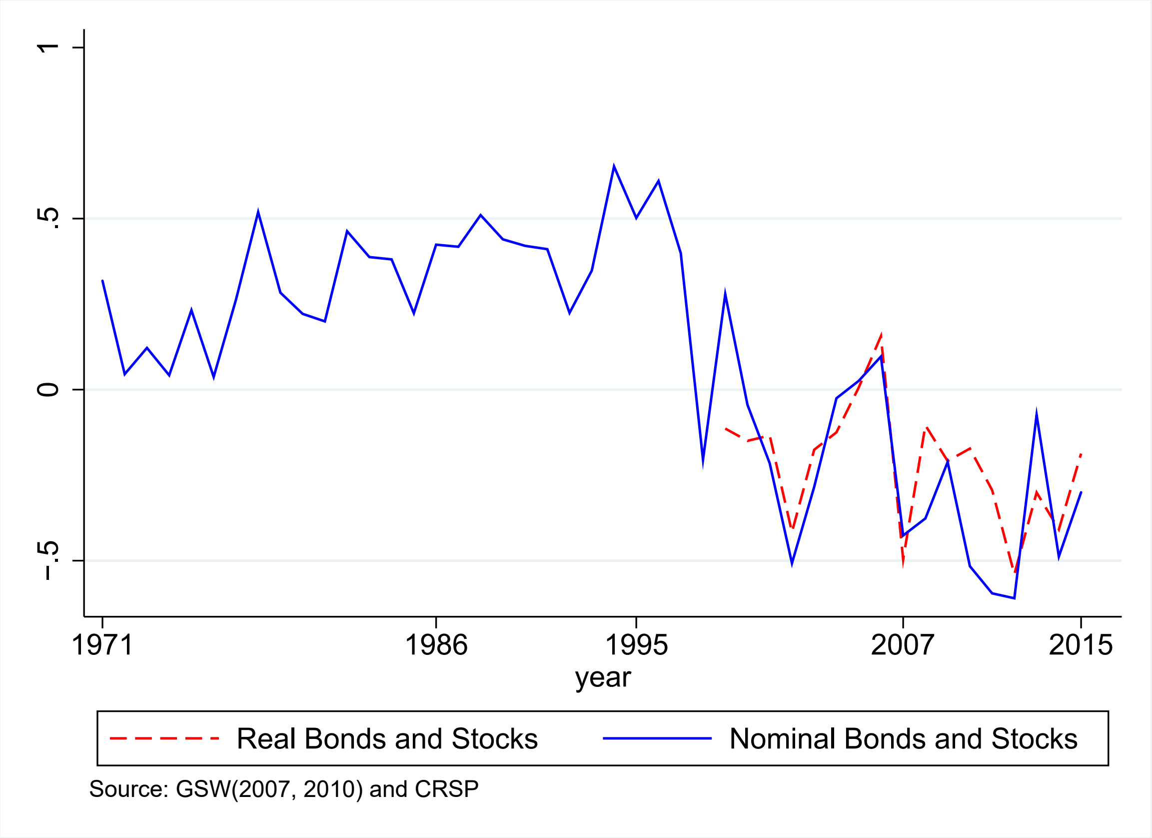
This table reports the point estimator and its standard deviation for each parameter of the volatility process. The table summarizes estimators for volatility parameters of cyclical component and distributions of the trend component. There are six parameters estimated. $\rho_{\sigma z}$ and $\rho_{\sigma g}$ denote the persistence of the volatility process. σ_z and σ_g denote the steady state log standard deviation of the cyclical shock and trend shock. $\eta_{\sigma z}$ and $\eta_{\sigma g}$ denote the standard deviation of shocks to the volatility process. The standard error estimates for elements of the parameter vector can be obtained from the outer product of the first derivative of log-likelihood function

Table 1.4: Prior and Posterior Distributions of Volatility Processes Parameters

Parameter	Prior distribution			Posterior Distribution			
	Distribution	Mean	Std.Dev.	Mean	Std.Dev.	5%	95%
Cyclical Shock							
$\rho_{\sigma z}$	Beta	0.9	0.1	0.998	0.0028	0.988	0.999
σ_z	Uniform	-5.0	2.0	-4.27	2.165	-7.49	-0.86
$\eta_{\sigma z}$	Gamma	0.2	0.1	0.076	0.027	0.032	0.134
Trend Shock							
$\rho_{\sigma g}$	Beta	0.9	0.1	0.999	0.0030	0.989	0.999
σ_g	Uniform	-5.0	2.0	-6.81	2.168	-9.96	-3.33
$\eta_{\sigma g}$	Gamma	0.2	0.1	0.091	0.026	0.043	0.15

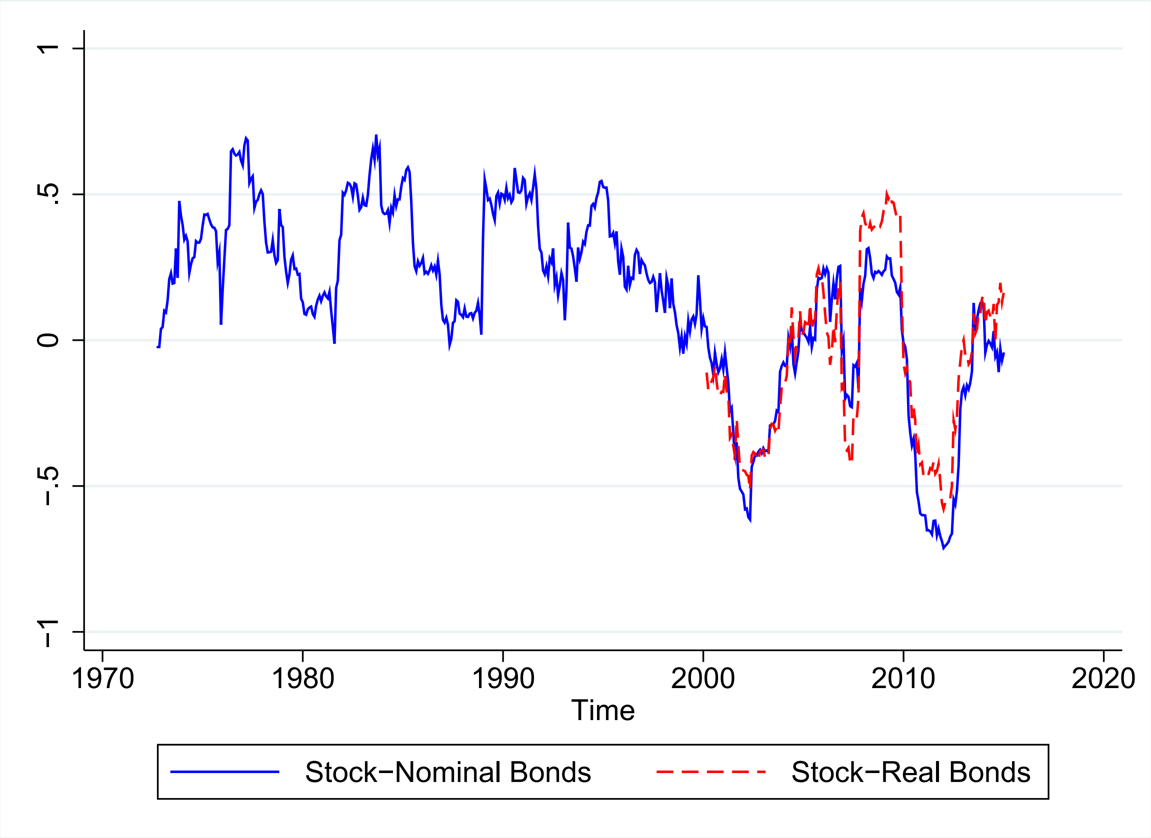
This table reports the prior and posterior distribution of parameters from the estimation of the model. The table summarizes distributions of volatility parameters of cyclical component and distributions of the trend component. There are six parameters estimated. $\rho_{\sigma z}$ and $\rho_{\sigma g}$ denote the persistence of the volatility process. σ_z and σ_g denote the steady state log standard deviation of the cyclical shock and trend shock. $\eta_{\sigma z}$ and $\eta_{\sigma g}$ denote the standard deviation of shocks to the volatility process.

Figure 1.1: Realized Stock-Bond Correlation for U.S.



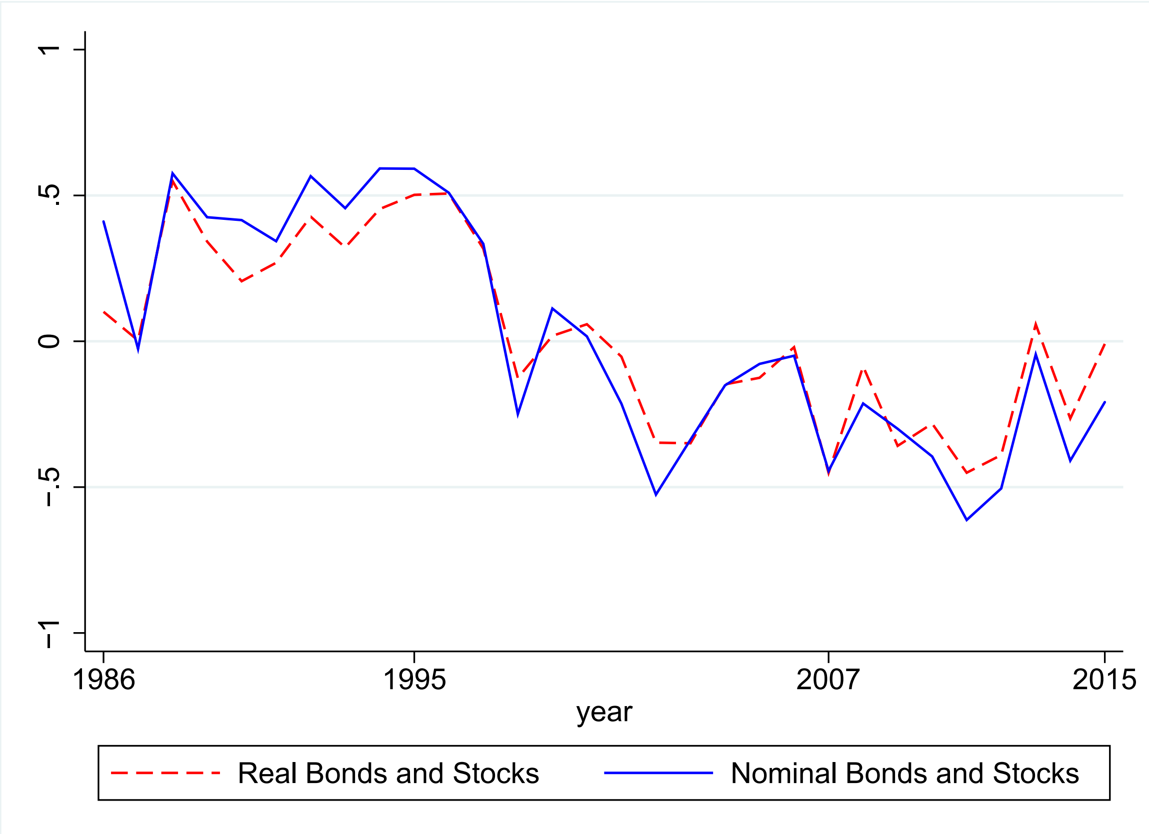
This figure graphs realized quarterly correlations measured using daily returns for nominal and real bonds in U.S. The data used for real bonds, which are known as the TIPs, starts at 1998.

Figure 1.2: Moving Stock-Bond Correlation for U.S.



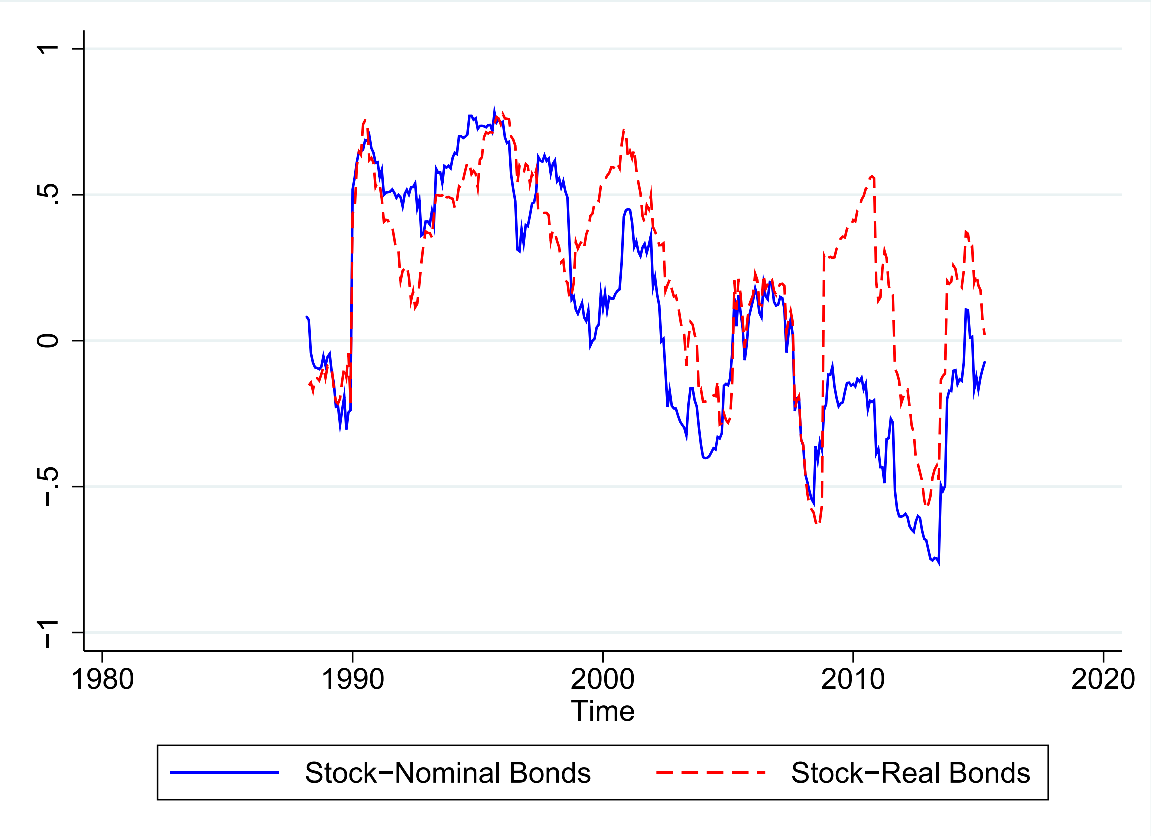
This figure displays correlations produced using monthly returns for nominal and real bonds in U.S. The estimate for month t is the sample correlation of the 25 returns for months $t-12$ through $t+12$.

Figure 1.3: Realized Stock-Bond Correlation for U.K.



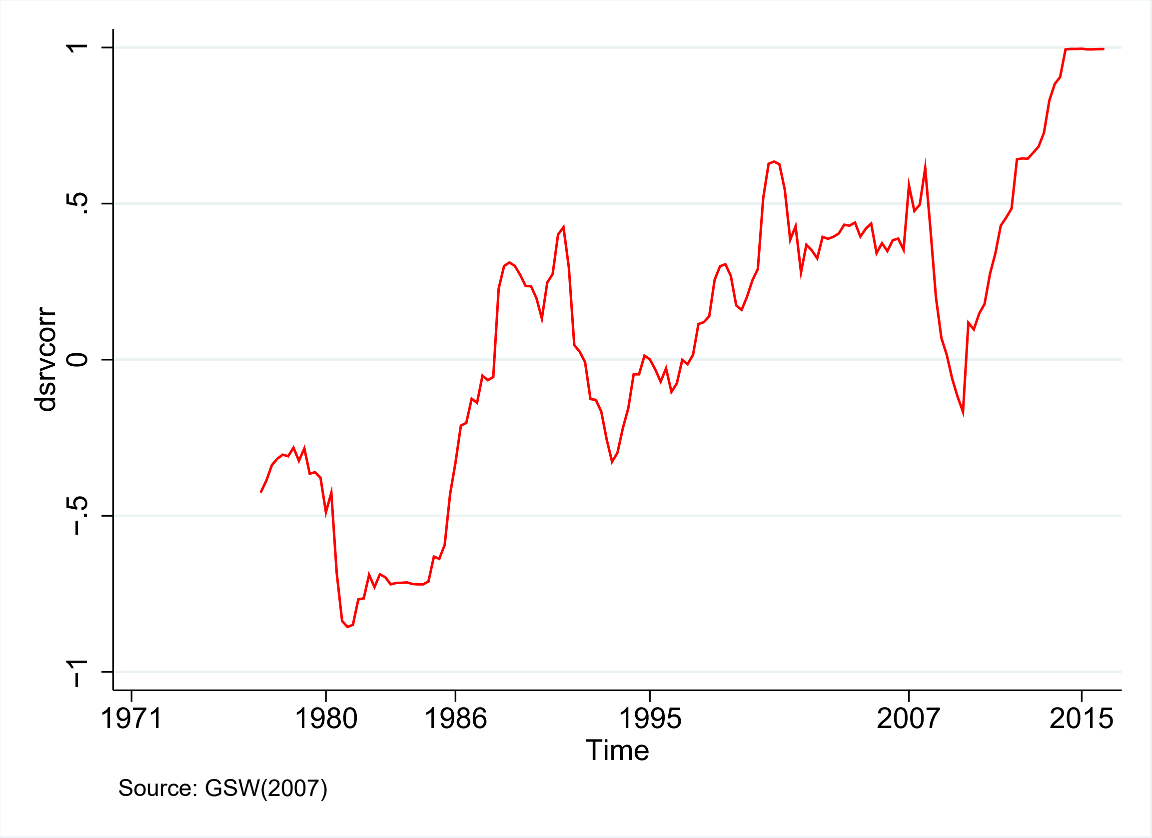
This figure graphs realized yearly correlations measured using daily returns for nominal and real bonds in UK. The data sample is from January 1986 to December 2015.

Figure 1.4: Moving Stock-Bond Correlation for U.K.



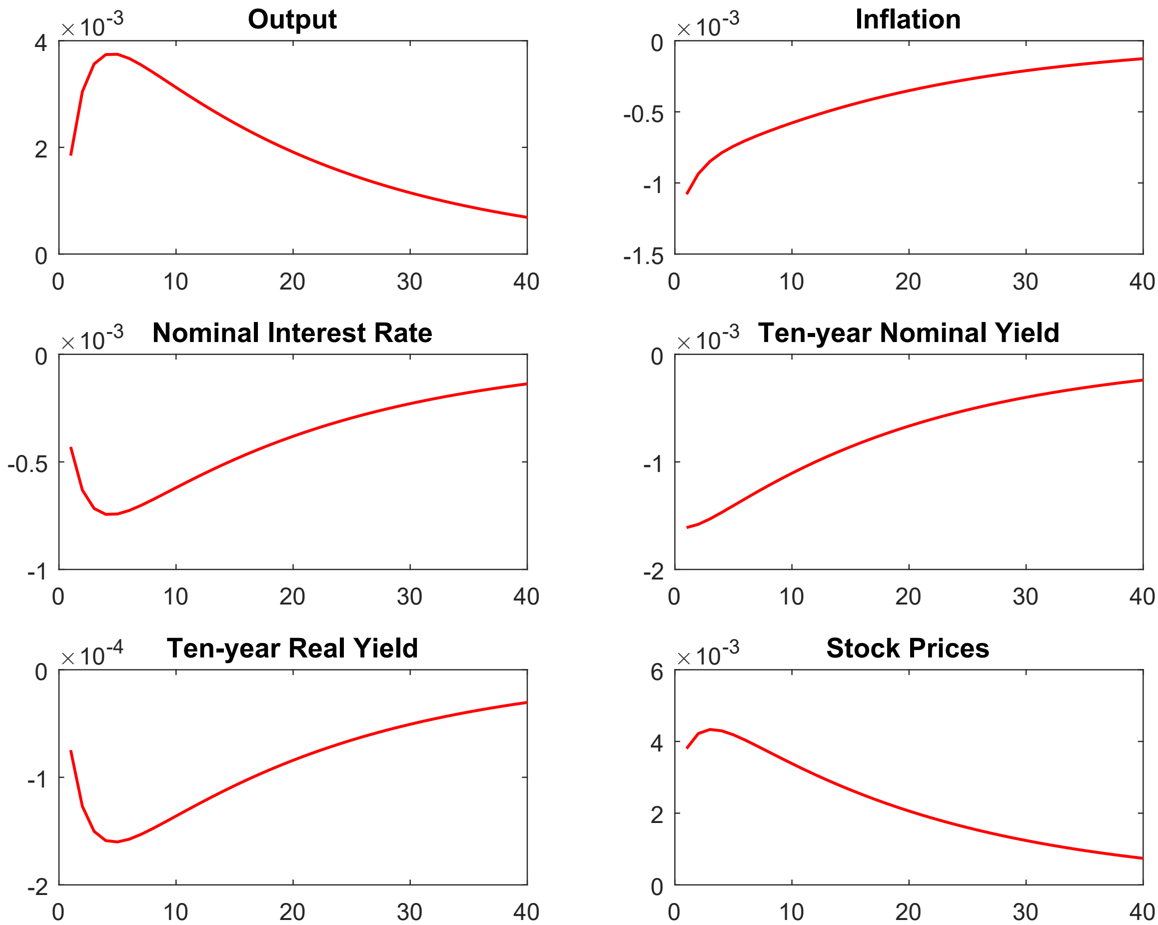
This figure displays correlations produced using monthly returns for nominal and real bonds in U.K. The estimate for month t is the sample correlation of the 25 returns for months $t - 12$ through $t + 12$.

Figure 1.5: Correlation between Slope Changes and Yields Changes



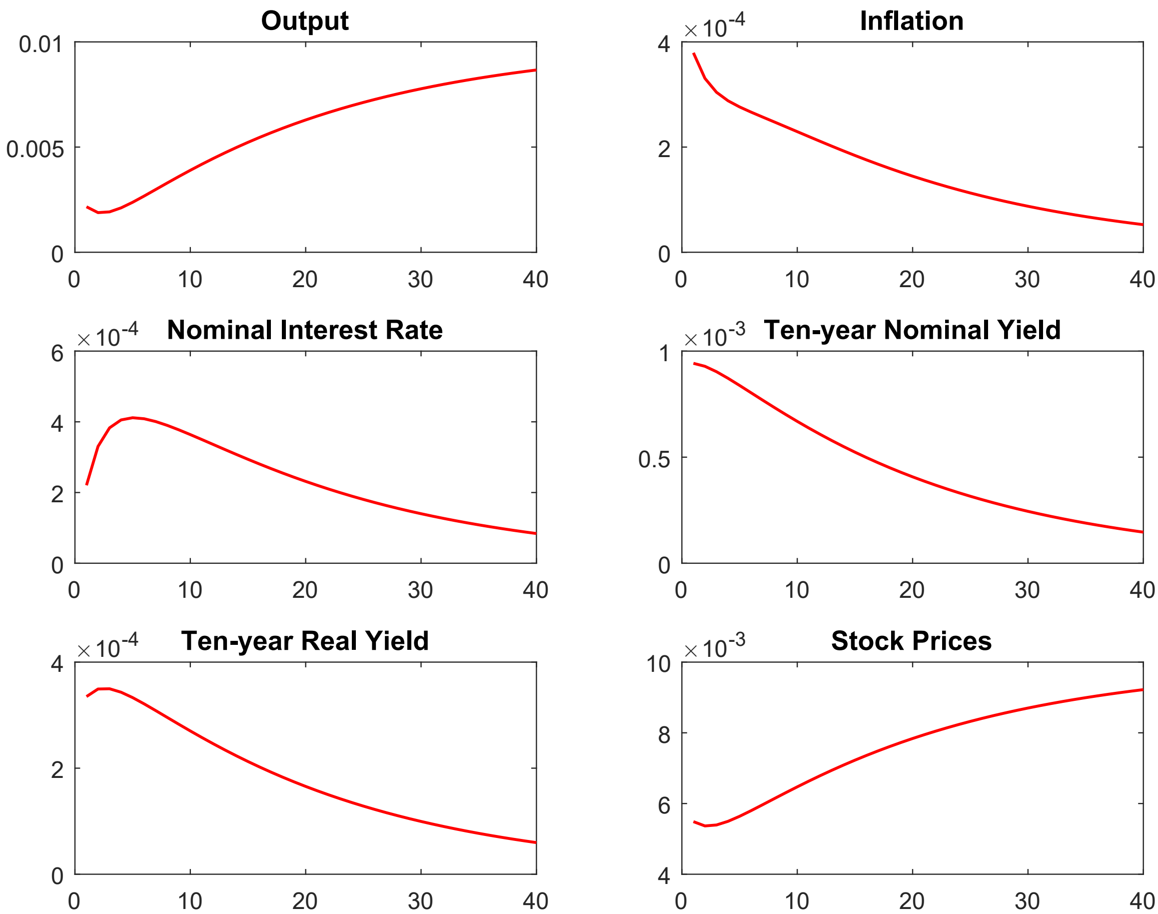
This figure graphs the 5-year moving quarterly correlations between changes in 5-year bond yield and changes in yield curve slope . The slope is measured by the 5-year yield less the three-month bill rate.

Figure 1.6: Impulse Response Functions for Level Shocks



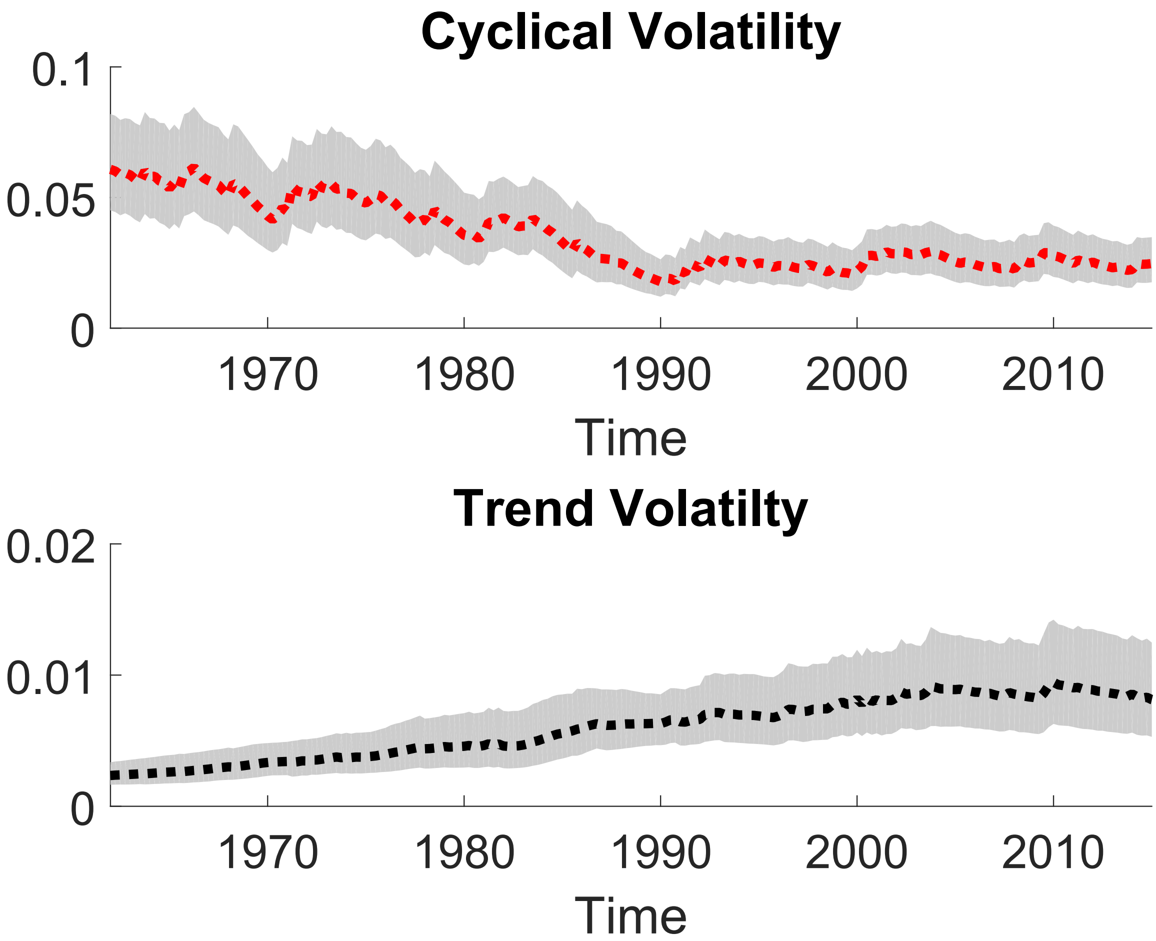
This figure shows average simulated impulse responses to a one standard deviation level technology shock for the output, inflation, the nominal interest rate, the nominal and real 10-year yields, and the stock prices.

Figure 1.7: Impulse Response Functions for Trend Shocks



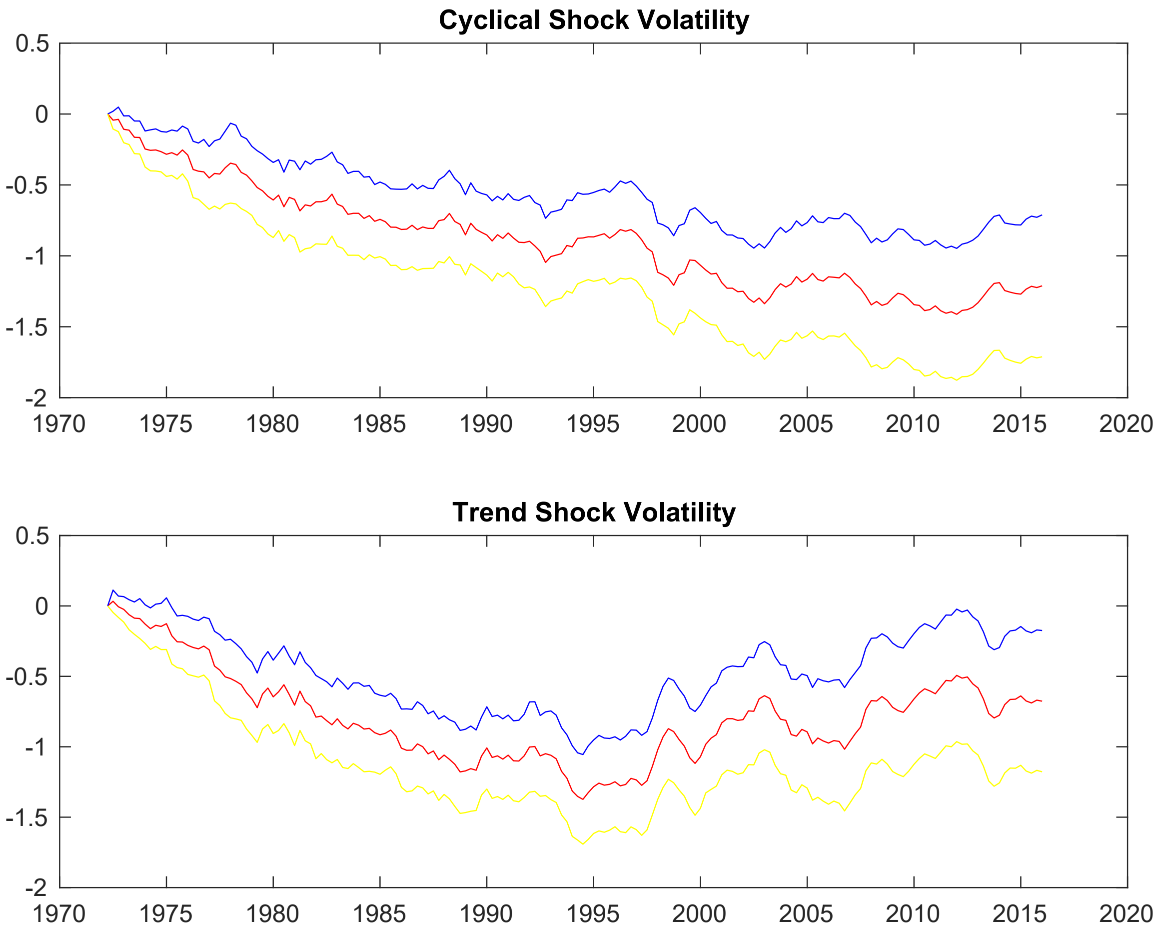
This figure shows average simulated impulse responses to a one standard deviation level technology shock for the output, inflation, the nominal interest rate, the nominal and real 10-year yields, and the stock prices.

Figure 1.8: Cyclical and Trend Shocks Volatility Estimated from Aggregate Productivity Growth



This figure shows the cyclical and trend volatility filter obtained from the productivity series. The shaded area represents the respective confidence interval.

Figure 1.9: Cyclical and Trend Volatility Estimated Using Both Productivity Growth and Stock-Bond Correlation



This figure shows the mean($\pm$ S.D.) of smoothed path of cyclical and trend volatility. The estimates are obtained using productivity growth and realized stock-bond correlation as observables.

Chapter 2

Idiosyncratic Volatility, Firm Investment and Capital Accumulation

2.1 Introduction

The productivity of a typical firm varies over time, reacting to both economy-wide and firm-specific productivity shocks. Conditional volatilities of these shocks also vary over time. Researchers following the seminal contribution of Bloom (2009) attempt to understand both the dynamics of these conditional volatilities and their affect on the aggregate economy.

This chapter first develops empirical methods to robustly quantify variations in the volatility of firm productivity shocks. I document empirical evidence of an upward trend in the volatility of firm-level productivity shocks. The trend remains robust after controlling for composition change of the data sample. Moreover, the upward trend in the volatility of firm-level productivity shocks is even stronger for firms that are younger, smaller, and in the technology sector. Thus, I argue that the upward trend in volatility is likely reflecting the fundamental change of the economy. This finding contributes to the literature on firm-level risk. While Campbell et al. (2001) and Comin and Mulani (2006) discover the upward trend in the stock return and sales growth volatility, my paper provides evidence that the rise in the firm-level risk may be driven by the increase in the volatility of productivity shocks.

Bloom et al. (2012) use the confidential Census Bureau data to measure the volatility of aggregate and idiosyncratic productivity shocks and find that both the aggregate and idiosyncratic volatility are countercyclical at the business cycle frequency. They measure the volatility of idiosyncratic productivity shocks as the cross-sectional dispersion of firm productivity shocks. Bachmann and Bayer (2013) also measure the volatility of firm productivity shocks using German firm-level data and study whether the volatility of firm level productivity shock is a major source of business cycle fluctuations.

My empirical analysis complements Bloom et al. (2012) in several ways. First, I robustly quantify the dynamics of the idiosyncratic volatility, controlling for the composition change of data sample and other firm characteristics. Second, I accommodate the estimation of firm production functions by employing the widely used Compustat database. In addition, I focus on the variation in idiosyncratic volatility at longer frequencies: the upward trend over the last five decades.

Existing papers, such as Bloom (2009) and Bachmann and Bayer (2013), have studied the short-run impact of idiosyncratic productivity shocks through dynamic models of firm investment. Their analysis focuses on the business cycle frequency variations. The main insight is that higher volatility increases the real option value of waiting and firms thus delay their investment. And the real option channel can operate through irreversibility and non-convex adjustment costs of investment. Because these two factors qualitatively operates through the same real option channel, I focus on the irreversibility of investment for tractability reason. Another noteworthy paper in the investment literature is Khan and Thomas (2008). They study a model of lumpy investment caused by fixed adjustment cost wherein firms face shocks to the level of productivity. Khan and Thomas (2008) argue the general equilibrium effect of interest rates on investment is large enough to offset the partial equilibrium investment behaviors due to nonconvex adjustment costs. Even though my model studies the impact of changes to idiosyncratic volatility, it remains important to investigate through a general equilibrium model.

To quantitatively study the aggregate consequences of such changes on firm investment and capital accumulation, I build a tractable general equilibrium model with firm heterogeneity. In the model, there is an intertemporal optimizing representative consumer and a continuum of firms differing in their productivity. A crucial feature of the model is that investment is irreversible at the firm level. I find that the increase in idiosyncratic volatility has a strong negative effect on the long-run investment and capital accumulation. In addition, the short-run impact of the volatility of firm productivity shocks operates mainly through two channels. The first impact is the partial equilibrium real option effect, as in Bloom (2009). When the firm-level productivity shock volatility increases, the option value of waiting rises and firms delay their investments. The reallocation of capital to the most productive units thus stalls. However, the decrease in investments corresponds to expected future decline in consumption growth. The standard consumption Euler equation would predict lower real interest rates in equilibrium. The decrease in interest rates partially offset the partial equilibrium effect.

My model analysis contributes to the literature in a few aspects. First, the model is based on the continuous-time firm investment model of Bertola and Caballero (1994). The key distinction is my model allows for more general dynamics for firm productivity and endogenizes aggregate interest rates in equilibrium. Second, Bloom (2009) and Bloom et al. (2012) only study the impact of shock idiosyncratic volatility at the business cycle frequency, whereas I focus on both the short-run and long-run impact of changes in idiosyncratic volatility.

2.2 Empirical Results

The section develops measures for the volatility of firm-specific productivity shocks and explores potential causes for its variation over time.

2.2.1 Measuring Firm-specific Productivity Innovation

The benchmark proxy to capture the volatility of firm-specific innovations is the cross-sectional dispersion of future firm-specific productivity innovations. Following Bloom et al. (2012), idiosyncratic productivity is measured by firm-specific Solow residual. The log TFP innovations ($\epsilon_{i,t}$) are estimated based on the following first order autoregressive equation about log productivity ($\omega_{i,t}$).

$$\hat{\omega}_{i,t+1} = \rho_{\omega}\hat{\omega}_{i,t} + \mu_i + \lambda_{t+1} + \epsilon_{i,t+1} \tag{2.1}$$

where $\hat{\omega}_{i,t}$ denotes the estimated log TFP (Total Factor Productivity). The benchmark idiosyncratic volatility measure $\sigma_{\epsilon,t}$ is defined to be standard deviation of firm-specific TFP shocks $\epsilon_{i,t}$ across firms at a given time t.

The specification controls for the firm fixed effect: μ_i and the time fixed effect: λ_t. The log firm level TFP is estimated for a panel of firms using data from Compustat. The data spans annually from 1963 to 2015. The method of estimating firm-level productivity adopts from Olley and Pakes (1996), which has been used by Imrohoroglu and Tüzel (2014) recently. This semi-parametric method is advocated because it is able to control for simultaneity and selection bias. A selection problem is generated by the relationship between productivity and the shutdown decision, and a simultaneity problem is produced by the relationship between productivity and input demands. The details of this estimation method are provided in the appendix. Figure 2.1 plots the time-series of the volatility of firm-level productivity shocks. The underlying data frequency is annual.

2.2.2 The Dynamics of Idiosyncratic Volatility

In the previous section, I don't make any functional assumption on the time-series dynamics of the volatility of firm productivity shocks. The methods build on the assumption that productivity shocks across firms are independent of each other, even though the volatility of firm productivity shocks can vary over time. At each point in time, we have a large of sample of firm specific productivity shocks. Therefore, the cross-sectional dispersion of firm level productivity shocks is a valid estimator for firm specific shock volatility in the time dimension. Besides, when the number of firms in the cross section gets large as here using the Compustat dataset, the estimator becomes an accurate proxy for idiosyncratic volatility. Table 2.1 report the time-series properties of the estimator of the idiosyncratic volatility at the annual frequency. The idiosyncratic volatility is very persistent with an estimated AR(1) coefficient of 0.91

and standard deviation of 0.05. Therefore, I cannot reject the hypothesis that shocks to idiosyncratic volatility can be permanent. I also consider the properties of changes to idiosyncratic volatility. I find the changes have an AR(1) coefficient of only -0.002. Thus, changes to idiosyncratic volatilities are very persistent and therefore could have important economics implications.

2.2.3 An Alternative Way to Measure Idiosyncratic Volatility of Productivity Shocks

An relevant question is whether the change of idiosyncratic volatility of productivity shocks measured in Section 2.2 are due to changing characteristics of the data sample. One way to control for composition effect is to look at changes in the volatility of productivity shocks at the firm level. For a given firm i with data for date $t-1$, t, $t+1$. I use the standard cross-sectional regression approach Olley and Pakes (1996) to calculate the productivity residuals for firm i in year t and $t+1$. Squaring them and taking the difference produce a (very noisy) measure of the change in firm i's volatility of idiosyncratic productivity shocks from t to $t+1$. Let $\Delta Vol_{i,t+1} \equiv \epsilon^2_{i,t+1} - \epsilon^2_{i,t}$ denotes this change. For each date $t+1$, I calculate $\Delta Vol_{t+1,EW}$: the equal-weighted mean of the change of volatility across all firms with non-missing $\Delta Vol_{i,t+1}$ or $\Delta Vol_{t+1,VW}$: value-weighted mean using market equity value at time t. An advantage of using value-weighted mean is that it would prevent the estimates from biased towards productivity shocks of a large number of small firms. The last step is to keep track of the level of productivity from the change of volatility over time. Let $Vol_{t,EW} \equiv \sum_{s=1}^{t} Vol_{s,EW}$ and $Vol_{t,VW} \equiv \sum_{s=1}^{t} Vol_{s,VW}$ denote these measures for the level of idiosyncratic volatility.

Figure 2.2 plots the time-series of these estimates from 1963 to 2015. It is clear from Figure 2.1 and 2.2 that there exists a robust upward trend for the level of idiosyncratic volatility. The value-weighted measure of idiosyncratic volatility is in general smaller than the equal-weighted one. The reason is likely that weighting by market capitalization downplays the increase in idiosyncratic volatility of small firms. Both the equal-weighted and value-weighted measures are strongly correlated with the cross-sectional dispersion measure $\sigma_{\epsilon,t}$ defined in Equation (2.1). The equal weighted measure has a correlation coefficient of 0.90 with respect to $\sigma_{\epsilon,t}$, while the value-weighted measure is significantly correlated with $\sigma_{\epsilon,t}$ with a coefficient of 0.78. Therefore, it is plausible to assert that changes in idiosyncratic volatility are not driven by the composition change of the Compustat data sample.

2.2.4 Rolling Window Measure of Idiosyncratic Volatility

Another way of measuring the volatility inherent in the firm's environment is by focusing on the time series. Formally, I consider the rolling time series for the volatility of $\epsilon_{i,t}$ as

$$Vol_{i,R} = \sqrt{\frac{\sum_{\tau=t-9}^{t}(\epsilon_\tau - \bar{\epsilon}_t)^2}{10}} \tag{2.2}$$

where $\bar{\epsilon}_t \equiv \sum_{\tau=t-9}^{t} \epsilon_\tau$. This measure could be more appealing in that it is less likely to be affected by the composition effect. When computing the standard deviation in the times series, I remove the average growth rate for the firm in the window, and in effect control for firm-specific aspects that affect the growth rate of productivity. These aspects, however, potentially show up in the cross-sectional measure and may be the medium through which a compositional bias operates. These standard deviations are then averaged across all the firms in a year to arrive a the average volatility for every year. As illustrated in Figure 2.3, volatility at the firm level exhibits a significant upward trend. In order to build a more representative measure, the standard deviations are weighted using the firm's market equity in a given year. Even though the trend become flatter using the value weighted measure, the volatility of firm productivity has increased more than 100% from the early 1960s to 2000s.

2.2.5 Controlling for Firm Size, Age and Sectors

I have presented three different measures for idiosyncratic productivity shocks volatility. Since the time series of all measures display the same upward trend, I focus on the first measure: the cross-sectional dispersion $\sigma_{\epsilon,t}$ hereafter. Figure 2.4

exhibits the time-series of the volatility for firms with different sizes. In each year, I divide firms into three groups based on their market capitalization. We can see that the upward trend in volatility holds for firms in different size groups. Relatively, the trend increase is stronger for small firms and weaker for large firms.

I conduct a similar exercise for firms in different age groups. Figure 2.5 shows that younger firms have a stronger trend increase in the productivity shocks volatility, while older firms go through a relatively smaller increase. The volatility for younger firms increases from 0.1 in 1960s to more than 0.4 in early 2000s. Older firms witnessed a comparatively milder increase in volatility, which rises from 0.1 in 1960s to more than 0.3 in early 2000s.

I also examine four main industries in this paper: consumer goods, manufacturing, health products and information, computer and technology industries. The classification of consumer goods, manufacturing, and health products industries are taken from Fama-French 5-industry classification.[1] The information, computer and technology industry classification is from the BEA Industry Economic Accounts, which consists of computer and electronics products, publishing industries (including software), information and data processing services, and computer systems design and related services. The patterns in the Figure 2.6 are intriguing. The information technology sector witnesses the strongest increase in productivity shock volatility. The peak volatility is 0.54 in year 2001 while the highest volatility for across all firms is 0.38

[1] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/Data_Library/changes_ind.html

at the same year. The consumer goods sector takes the smallest increase in volatility with the peak of 0.28 of in year 2012. Therefore, the dynamics of productivity shocks volatility exhibit a significant degree of heterogeneity. The increase of volatility is stronger for firms which are younger, smaller and making more investments, such as firms in the technology sector.

2.2.6 The Rise of Idiosyncratic Volatility

A striking pattern in previous figures is the substantial and robust rise of the volatility of firm-level productivity shocks over the last fifty years.This finding contributes to the literature on firm-level volatility dynamics. Campbell et al. (2001) discovers the upward trend in the cross-sectional dispersion of the component of returns that is unrelated to the average return in the four-digit sector. Comin and Mulani (2006) documents the same upward trend in firm sales growth volatility. My paper establishes that the volatility of productivity shocks has risen substantially, which suggests that the upward trend of firm risk may be firm fundamental driven. While Bloom et al. (2012) also measures the volatility of TFP shocks, their approach is mostly suited for the confidential Census Bureau data. My empirical approach is based on the Compusutat/CRSP dataset, which is more available for researchers. Besides, I conduct further robustness control and estimate the production functions. Therefore, my empirical analysis can be viewed as a complement to theirs.

I have reported idiosyncratic productivity volatility for firms with different size, age and sectors. Younger and smaller firms experience larger hikes in the volatility of productivity shocks. Firms in the information, computer and technology industry experience the largest increase in productivity shocks volatility, while consumer goods sector firms experience a relatively mild growth. Even though firms vary in the degree of growth in productivity shock volatility, the upward increase in volatility is robustly significant across firms. This highlights the potential aggregate consequences of such dynamics of volatility on aggregate investment and capital accumulation. To further quantitatively shed light on this question, it is therefore important to investigate through an economic model with emphasis on firm investment and volatility dynamics.

2.3 The Model

In this section, I analyze the quantitative impact of variation in idiosyncratic volatility within a continuous time dynamic general equilibrium model. Specifically, I consider an economy with heterogeneous firms that make irreversible investments to produce a final good. The economy consists of a representative household and a continuum of firms with a unit mass. Assume that there is no aggregate uncertainty and that firms face idiosyncratic productivity shocks. By the law of larger numbers, all aggregate quantities are deterministic over time, although each firm is still exposed to idiosyncratic uncertainty. Firms that adjust their capital stock incur adjustment costs.

To study the long-run impact of idiosyncratic volatility variation, I investigate the steady states properties of the model under different levels of idiosyncratic volatility. In particular, I focus on investment and capital accumulation. I also examine the short-run dynamics of the model through the transition path of the economy from one steady state to a new one, which is caused by an unexpected increase in the idiosyncratic productivity shock volatility [2] [3].

---

[2]Compared with a fully dynamic model in which the idiosyncratic volatility follows a stationary process and agents know that the volatility of idiosyncratic volatility is time-varying, my model would tend to overpredict the effect of an increase in the idiosyncratic volatility. In my setup, agents don't have precautionary motives against changes of volatility and assume that the change of volatility is permanent.

[3]It is well known from papers such as Krusell and Smith (1998) and Khan and Thomas (2008) that the cross-sectional distribution of firm capital accumulation becomes infinite dimension state variables to keep track of in heterogeneous agents model with aggregate shocks. The absence of aggregate shocks in this paper significantly simplifies the state space and the solution of the model.

2.3.1 Households

The economy is populated by a continuum of identical households. They have preferences with the same discount rate ρ, and elasticity of intertemporal substitution (EIS) ψ^{-1}. They are defined following, for example, Cass (1965)

$$U_t = \mathbb{E}_t \left[\int_t^{\infty} f(C_u)du \right], \quad f(C_u) = \rho \frac{C^{1-\psi}}{1-\psi} \tag{2.3}$$

The term C_t is the consumption rate at time t. Let W denotes the wealth of the representative agent and $J(W)$ the value function. In equilibrium, it must be the case that $J(W_t) = U_t$. To solve for the household value function, consider the Hamilton-Jacobi-Bellman (HJB) equation for an investor who allocates wealth between the claim to all dividends (stock market) in the economy and the risk-free asset. Since there is no aggregate risk in the economy, wealth follows the process

$$dW_t = (r_t W_t - C_t)dt$$

The solution to the representative agent's consumption and portfolio choice problem is given by the following HJB equation (Duffie and Epstein (1992))

$$0 = \max_{C_t} f(C_t) + J_W[r_t W - C_t] \tag{2.4}$$

Taking the first-order condition with respect to C, we have

$$f_C(C_t) - J_W = C_t^{-\psi} - J_W = 0$$

For further analysis, it is convenient to calculate the state-price density, which prices consumption goods in different states of the world. In particular, it can be shown (e.g., Duffie and Epstein (1992)) that the state-price density Λ_t equals to

$$\Lambda_t = \exp(-\rho t) C_t^{-\psi} \tag{2.5}$$

By the law of one price, the state price density evolves following

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t dt$$

where r_t is the real risk-free interest rate in the economy. Since there is no aggregate uncertainty in the model, aggregate variables including the risk-free rate are deterministic over time. The interest rates could vary when the economy is making transitions to new steady states.

2.3.2 Firms

There is a continuum of firms indexed by $i \in [0, 1]$ with idiosyncratic productivity $Z_{i,t}$. Productivity $Z_{i,t}$ follows some Markov process. The productivity of each firm is independent of each other. Each firm produces goods with productivity $Z_{i,t}$. This standard setup for firm heterogeneity can be seen, for example, at Khan and Thomas (2008). The firm-level productivity follows a diffusion process

$$Z_{i,t} = \mu(Z_{i,t})dt + \sigma_t dW_{i,t} \tag{2.6}$$

It is important to note that the firm level productivity shock volatility can only change deterministically over time. This assumption is only made for tractability reason.

Let K_t denotes the firm's capital stock and the process L_t represents the cumulative gross investment up to date t. Investment is often irreversible in that installed capital has little or no value unless used in production. Following Bertola and Caballero (1994), I assume that investment at the firm level is irreversible in the sense that the capital has no resale value. Thus, it is never worthwhile for firms to disinvest and the gross investment process L_t is nondecreasing over time. Ramey and Shapiro (2001) suggests that this assumption is realistic, at least for some industries. Capital depreciates at the constant rate $\delta \geq 0$, so the stochastic process for the capital stock of firm i is

$$dK_{i,t} = dL_{i,t} - \delta K_{i,t}dt \tag{2.7}$$

Summing over all firms in the economy, the law of motion for the aggregate capital stock is

$$dK_t = \int_i dK_{i,t} - \delta K_t dt$$

The firm's objective is to maximize expected discounted profits. Hence its problem is [4]

$$V(K_t, Z_t) = \max_{L(t+u), u \geq 0} \mathbb{E}_t \left[\int_0^{\infty} \frac{\Lambda_{t+u}}{\Lambda_t} \{ \Pi(K_{t+u}, Z_{t+u}) du - dL(t+u) \} \right] \tag{2.8}$$

Under standard techniques (see e.g., Stokey (2008)), it is possible to show that the optimal investment policy is defined by a threshold function $b(Z)$. If $K < b(Z)$ the firm makes discrete investment of size $b(Z) - K$, so below the threshold the value function is

$$V(K, Z) = V((b(Z), Z)) + b(Z) - K, \quad K < b(Z) \tag{2.9}$$

The region above $b(Z)$ is the inaction region. In this region, the value function satisfies the Hamilton-Jacobi-Bellman (HJB) equation associated with the optimization

[4] I suppress the firm specific subscript i for simplicity.

problem (2.10)

$$r_t V(K_{i,t}, Z_{i,t}) = [\Pi(K_{i,t}, Z_{i,t}) - \delta K_{i,t} V_K(K_{i,t}, Z_{i,t})] + \mu(Z_{i,t}) V_Z(K_{i,t}, Z_{i,t}) \\ + \frac{1}{2}\sigma(Z_{i,t})^2 V_{ZZ} \quad \text{if} \quad K \geq b(Z) \tag{2.10}$$

The term on the left side of (2.10) denotes the expected interest of investing at time t. The first term on the right hand side gives the expected cash flow. The second term on the right gives the drift and volatility effects of productivity change on $V(K, Z)$.

2.3.3 Heterogeneity and Aggregation

In order to solve for the equilibrium, it is necessary to keep track of the cross-sectional distribution of firm capital stock to characterize the dynamics of the aggregate state of the economy. Firms in the economy are indexed by their productivity types Z and capital stock K. At each point in time t, it is important to keep track of the joint distribution of capital and productivity: $g_t(K, Z)$. The corresponding marginal distributions are denoted by $\phi_t(K)$ and $\psi_t(Z)$. It is straightforward to define the aggregate capital stock K_t as

$$K_t = \int_K \int_Z K g_t(K, Z) dK dZ \tag{2.11}$$

The total output in the economy Π_t then follows

$$\Pi_t = \int_K \int_Z K^\alpha Z^{1-\alpha} g_t(K,Z) dK dZ \tag{2.12}$$

Similarly, the aggregate investment I_t is represented by

$$I_t = \int_{K<b(Z)} \int_Z (b(Z) - K) g_t(K,Z) dK dZ \tag{2.13}$$

The next proposition is the main tool to characterize the evolution of the cross-sectional distribution of and capital stock K and productivity Z.

Proposition 1 ***(The Evolution of Cross-Sectional Distribution)*** *The cross-sectional distribution $g_t(K,Z)$ obeys the second order partial differential equation*

$$\partial_t g_t(K,Z) = \partial_K \left(\max(b(Z) - K, 0) g_t(K,Z)\right) - \partial_Z \left(\mu(Z) g_t(K,Z)\right) + \frac{1}{2}\partial_{ZZ}\left(\sigma^2(Z) g_t(K,Z)\right) \tag{2.14}$$

The partial differential equation is mathematically similar to the Kolmogorov Forward equation to keep track of the distributions of diffusion processes. This method to keep track of the cross-sectional distribution of firm capital stock is based on Achdou et al. (2014). While they study the heterogeneity on the household side, I focus on the heterogeneity at the firm level. [5]

[5] There are unfortunately no easy explanations for the proposition. An illustrative example is when Z is a constant. The last two terms on the RHS become zero. The first term on the RHS keeps track of the rate of change of K.

2.3.4 Equilibrium

With the characterization of the optimal firm policies and aggregate quantities complete, I now state the definition of the competitive general equilibrium.

Definition 1 ***(Competitive Equilibrium)*** *A competitive equilibrium is a set of processes: aggregate consumption C_t, the state price density π_t, aggregate capital stock K_t; and a set of stochastic processes for each firm $i \in \mathbb{I}$: investment I_i, capital stock $K_{i,t}$, output $\Pi_{i,t}$ such that*

(1) The representative consumer and each firm solve their problems taking aggregate conditions as given.

(2) Market clearing:

$$\int_i K_{i,t} di = K_t$$
$$\int_i \Pi_{i,t} di = \Pi_t$$
$$C_t + I_t = \Pi_t$$

(3) Aggregate capital stock satisfies the law of motion, starting from K_0:

$$dK_t = \int_i I_{i,t} dt - \delta K_t$$

The market clearing conditions for the consumption goods and capital market are standard. An important class of the equilibrium is the steady state of the economy which is defined as follows.

Definition 2 ***(Steady State)*** *The steady state of the economy is characterized by a competitive equilibrium path in which*

(1) The aggregate consumption growth rate and the risk-free rate r_t are constant over time.

(2) the cross-sectional distribution of firm capital stock is invariant over time.

In the steady state of the model, the stationary cross-sectional distribution of endogenous state variables has been reached. By the law of large numbers, economic aggregates are constants over time.

I now state the exact formulation of the equilibrium conditions

Proposition 2 ***(Equilibrium Conditions)*** *The equilibrium is characterized by the following partial differential equation systems*

$$0 = \max_{C_t} f(C_t, J(W_t)) + J_W[r_t W_t - C_t] \tag{2.15}$$

$$\begin{aligned} r_t V(K_{i,t}, Z_{i,t}) =& [\pi(K_{i,t}, Z_{i,t}) - \delta K_{i,t} V_K(K_{i,t}, Z_{i,t})] + \mu(Z_{i,t}) V_Z(K_{i,t}, Z_{i,t}) \\ &+ \frac{1}{2}\sigma(Z_{i,t})^2 V_{ZZ} \qquad \text{if} \quad K \geq b(Z) \end{aligned} \tag{2.16}$$

$$V(K, Z) = V((b(Z), Z)) + b(Z) - K \quad \text{if} \quad K < b(Z) \tag{2.17}$$

$$C_t + I_t = \Pi_t \tag{2.18}$$

$$\partial_t g_t(K, Z) = \partial_K \left(\max(b(Z) - K, 0) g_t(K, Z)\right) - \partial_Z \left(\mu(Z) g_t(K, Z)\right) + \frac{1}{2}\partial_{ZZ}\left(\sigma^2(Z) g_t(K, Z)\right) \tag{2.19}$$

2.4 Computing Algorithms

The section develops the numerical solution for the heterogeneous firms model using a finite difference method of partial differential equations.[6] The finite difference methods have been successfully used to value options and other derivative securities. To obtain valuation formula for warrants, Schwartz (1977) proposes to use the finite difference method to numerically solve the partial differential equations. Hull and White (1990) extends the standard finite difference method to price a wider class of derivative securities. The flexibility of the finite difference method facilitates the computation of my model equilibrium.

2.4.1 Computing Steady States

In this section, I describe how I calculate the stationary equilibria. Since the aggregate economy is stationary and there exists a representative agent, the stationary interest can be proven to be equal to ρ: the time preference rate.

1. Given the steady state interest rate $r = \rho$, solve the firm's HJB equation (2.10) using a finite difference method. Calculate the investment threshold $b(Z)$.

2. Given the investment threshold function $b(Z)$, solve the Kolmogorov Forward equation (2.14) using a finite difference method.

[6] The appendix establishes analytical solutions under specific functional assumptions about the productivity process. The analytical solutions are helpful to develop intuitions and verify numerical results of the model.

3. Given the cross-sectional distribution of $g(K, Z)$ and investment threshold $b(Z)$, compute the aggregate output Π_t and aggregate investment I_t. The aggregate consumption is then $C_t = \Pi_t - I_t$ by the market clearing condition.

4. Given the aggregate consumption C, solve the HJB equation of the representative household. Compute J_W and r and verify that $r = \rho$

2.4.2 Finite Difference Methods

For step 1, the finite difference method approximates the functions V at I grid points in capital K, $K_i, i = 1, ...I$ and J grid points in productivity dimension Z, Z_j, $j = 1, ...J$. I use equispaced grids denote by ΔK and ΔZ the distance between grid points, and use short-hand notation $V_{i,j} = V(K_i, Z_j)$. The derivative $\partial_K V_{i,j} = \partial_K V(K_i, Z_j)$ is approximated with either a forward or backward difference approximation

$$\partial_{K,F} V_{i,j} = \frac{V_{i+1,j} - V_{i,j}}{\Delta K} \tag{2.20}$$

$$\partial_{K,B} V_{i,j} = \frac{V_{i,j} - V_{i-1,j}}{\Delta K} \tag{2.21}$$

Similarly, the finite difference for the derivation in productivity Z follows

$$\partial_Z V_{i,j} = \frac{V_{i,j+1} - V_{i,j}}{\Delta Z} \tag{2.22}$$

$$\partial_{ZZ} V_{i,j} = \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{(\Delta Z)^2} \tag{2.23}$$

Let n denotes the number of iterations implemented to find the solution to the HJB equation. I use the following finite difference approximation to updates the value function $V^n(K, Z)$ (2.10)

$$\frac{V_{i,j}^{n+1} - V_{i,j}^{n}}{\Delta} + \rho V_{i,j}^{n+1} = \Pi_{i,j}^{n} - \delta K_i \partial_K V_{i,j}^{n+1} + \mu_j \partial_Z V_{i,j}^{n+1} + \frac{\sigma_j^2}{2} \partial_{ZZ} V_{i,j}^{n+1} \tag{2.24}$$

where the parameter Δ is the step size.

2.4.3 Upwind Scheme

The upwind scheme is to use the forward approximation whenever the drift of the state variable is positive and the backward difference approximation whenever it is negative. Since in the inaction region, the capital stock size is drifting down due to depreciation. I use the following finite difference approximation

$$\frac{V_{i,j}^{n+1} - V_{i,j}^{n}}{\Delta} + \rho V_{i,j}^{n+1} = \Pi_{i,j}^{n} - \delta K_i \partial_{K,B} V_{i,j}^{n+1} + \mu_j \partial_Z V_{i,j}^{n+1} + \frac{\sigma_j^2}{2} \partial_{ZZ} V_{i,j}^{n+1} \tag{2.25}$$

Substituting the definitions for finite differences above, we have

$$\frac{V_{i,j}^{n+1}-V_{i,j}^{n}}{\Delta}+\rho V_{i,j}^{n+1}=\Pi_{i,j}^{n}-\delta K_i\frac{V_{i,j}^{n+1}-V_{i-1,j}^{n+1}}{\Delta K}+\mu_j\frac{V_{i,j+1}^{n+1}-V_{i,j}^{n}}{\Delta Z}+\frac{\sigma_j^2}{2}\frac{V_{i,j+1}^{n+1}-2V_{i,j}^{n+1}+V_{i,j-1}^{n+1}}{(\Delta Z)^2} \tag{2.26}$$

The equation (2.26) constitutes a system of $I \times J$ linear equations and can be written in matrix forms in the following steps. Collecting terms with the same subscripts on the right hand side, we have

$$\begin{aligned}
&\frac{V_{i,j}^{n+1}-V_{i,j}^{n}}{\Delta}+\rho V_{i,j}^{n+1}=\Pi_{i,j}^{n}+x_{i,j}V_{i-1,j}^{n+1}+y_{i,j}V_{i,j}^{n+1}+\chi_j V_{i,j-1}^{n+1}+\zeta_j V_{i,j+1}^{n+1}\\
&x_{i,j}=\frac{\delta K_i}{\Delta K},\quad y_{i,j}=-x_{i,j},\quad \nu_j=-\frac{\mu_j}{\Delta Z}-\frac{\sigma_j^2}{(\Delta Z)^2}\\
&\chi_j=\frac{\sigma_j^2}{2(\Delta Z)^2},\quad \zeta_j=\frac{\mu_j}{\Delta Z}+\frac{\sigma_j^2}{2(\Delta Z)^2}
\end{aligned} \tag{2.27}$$

It is important to note that $x_{1,j}=0$ for all j because the size of the capital stock is bounded in the approximation schemes. At the boundaries of the productivity dimension j, the equation become

$$\frac{V_{i,1}^{n+1}-V_{i,1}^{n}}{\Delta}+\rho V_{i,1}^{n+1}=\Pi_{i,1}^{n}+x_{i,1}V_{i-1,1}^{n+1}+(y_{i,1}+\nu_1)V_{i,1}^{n+1}+\chi_1 V_{i,1}^{n+1}+\zeta_1 V_{i,2}^{n+1} \tag{2.28}$$

$$\frac{V_{i,J}^{n+1}-V_{i,J}^{n}}{\Delta}+\rho V_{i,J}^{n+1}=\Pi_{i,J}^{n}+x_{i,J}V_{i-1,J}^{n+1}+(y_{i,J}+\nu_J)V_{i,J}^{n+1}+\chi_J V_{i,J-1}^{n+1}+\zeta_J V_{i,J}^{n+1} \tag{2.29}$$

where I have used that $V_{i,0} = V_{i,1}$ and $V_{i,J} = V_{i,J+1}$. The equation (2.27) can be written in matrix notation as:

$$\frac{1}{\Delta}(V^{n+1} - V^n) + \rho V^{n+1} = \Pi^n + \mathbb{A}^n V^{n+1} \tag{2.30}$$

where V^n is a vector of length $I \times J$ with entries $(V_{1,1}, ..., V_{I,1}, ...V_{1,J}, ...V_{I,J})'$ and $\mathbb{A}^n = \mathbb{B}^n + \mathbb{C}$ where the $(I \times J) \times (I \times J)$ matrices $\mathbb{B}^n$ and $\mathbb{C}$ are defined as

$$\mathbb{B}^n = \begin{bmatrix}
y_{1,1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
x_{2,1} & y_{2,1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & x_{I,1} & y_{I,1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & y_{1,2} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & x_{2,2} & y_{2,2} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & x_{I,2} & y_{I,2} & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & y_{1,J} & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & x_{2,J} & y_{2,J} & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & x_{I,J} & y_{I,J}
\end{bmatrix} \tag{2.31}$$

$$
\mathbb{C} = \begin{bmatrix}
\nu_1+\chi_1 & 0 & \cdots & \cdots & \zeta_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \nu_1+\chi_1 & 0 & \ddots & \ddots & \zeta_1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \nu_1+\chi_1 & 0 & \ddots & \ddots & \zeta_1 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\chi_2 & \ddots & \ddots & 0 & \nu_2 & 0 & \ddots & \ddots & \zeta 2 & \ddots & \ddots & \ddots & \vdots \\
0 & \chi_2 & \ddots & \ddots & 0 & \nu_2 & 0 & \ddots & \ddots & \zeta 2 & \ddots & \ddots & \vdots \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 0 & \chi_2 & 0 & \ddots & 0 & \nu_2 & 0 & \ddots & \ddots & \zeta 2 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \chi_J & 0 & \ddots & \ddots & \nu_J+\chi_J & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \chi_J & 0 & \ddots & 0 & \nu_J+\chi_J & 0 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \chi_J & 0 & \cdots & 0 & \nu_J+\chi_J
\end{bmatrix} \tag{2.32}
$$

2.4.4 Kolmogorov Forward Equation

I now turn to the solution of the (2.19). The equation is discretized similar to the finite difference method used for the HJB equation. The technical details can be seen at Achdou et al. (2014).

2.4.5 Computing Transition Dynamics

I compute the transition dynamics using the following algorithm. Approximate the value function at N discrete points in the time dimension. Use the short-hand notation $v_{i,j}^n = V(K_i, Z_j, t^n))$. Guess a function $r^0(t)$, then for $\ell = 0, 1, 2..$ follow

(1) Given $r^\ell(t)$, solve the firm's HJB equation (2.10) with terminal condition $V^T(K, Z) = V(K, Z)$ backward in time to compute the time path of $V_{i,j}^n$. Also compute the implied investment threshold $b_t^\ell(Z)$.

(2) Given the investment threshold $b_t^\ell(Z)$, solve the Kolmogorov Forward equation with initial condition $g_0^t(K, Z) = g(K, Z)$ forward in time to compute the time path for $g^\ell(K, Z, t)$.

(3) Given $g^\ell(K, Z, t)$ and $b_t^\ell(Z)$, calculate aggregate investment I_t and output Π_t.

(4) Given $r^\ell(t)$, solve the representative's agent's HJB equation (2.4) with terminal condition $J^T(W_T)$. Compute the consumption C_ℓ^n.

(5) Given $b_t^\ell(Z)$, $g^\ell(K, Z, t)$ and C_ℓ^n, calculate the surplus

$$S^\ell(t) = \Pi_t - C_t - I_t$$

.

(6) Update $r^{\ell+1}(t) = r^\ell(t) - \xi \frac{dS^\ell(t)}{dt}$, where $\xi > 0$.

(6) Stop when $r^{\ell+1}$ is sufficiently close to $r^{\ell}(t)$.

2.5 Quantitative Implications of Idiosyncratic Volatility

The main purpose of this section is to illustrate the impact of idiosyncratic volatility on aggregate investment and capital allocation both in steady states and during transitions. I show that idiosyncratic volatility has important implications for the quantities in steady states and transition dynamics. Bloom et al. (2012) structurally estimate a dynamic general equilibrium model to study the impact of uncertainty shocks at the business cycle frequency. My paper differs from theirs in the focus on idiosyncratic volatility and long-run capital accumulation.

2.5.1 The Productivity Process

The framework laid out in the previous section works with a relatively general process. In the numerical analysis of this section, I consider the case that idiosyncratic productivities are following Ornstein-Uhlenbeck processes.

$$dZ_t = \theta(\mu - Z_t)dt + \sigma dW_t \tag{2.33}$$

where μ represents the mean value of productivity, σ is the degree of volatility and θ the rate by which these shocks dissipate and the productivity reverts towards the mean. An attractive feature of this process is that it is the exact continuous time counterpart to a discrete-time AR(1) process. Also I specify the profit function $\Pi(K_t, Z_t) = K_t^{\alpha} Z_t^{1-\alpha}$ to be of constant returns to scale in (K_t, Z_t).

As a brief aside, I would like to note that for alternatives to the Ornstein-Uhlenbeck process, the steady state cross-sectional distribution of investment rate may be actually solved in closed forms. I provide one example in which productivity follows geometric Brownian motion in the Appendix. In that case, there is a strictly negative relationship between the idiosyncratic volatility of productivity shocks and firm investment.

2.5.2 The Parameters of the Model

In the model, time is continuous and the length of unit interval corresponds to one year. I set α: the capital share in the production function to 0.33. The persistence parameter of productivity: θ equals 0.3, which corresponds to an annual first-order autocorrelation of productivity of 0.7. The depreciation rate is set to 0.1 in the annual sense. The time preference rate ρ equals to 0.05. These values are standard in the macroeconomic literature. The choice of intertemporal elasticity of substitution (IES) is subject to discretion. Hall (1988) estimates the IES to be well below 1. Bansal and Yaron (2004) argues that an IES of 2 is important to reconcile asset pricing moments.

I choose the inverse elasticity of intertemporal substitution to be 0.5 and 5 in my quantitative exercises.

In my quantitative exercises, I analyze steady states of the model with the volatility of productivity shocks $\sigma = 0.1$ and $\sigma = 0.2$. I also consider the transition path from the low volatility steady state to the high volatility one. These two values roughly correspond to the volatility of firm productivity shock in early 1960s and the more recent value. The values of the parameters are listed in Table 2.2. Since there are no aggregate shocks ex-ante, the aggregate consumption, investment and real interest rates are constants in the economy. The intertemporal elasticity of substitution plays no role in determining quantities in steady states.

2.5.3 The Effect of Idiosyncratic Volatility on Steady States

When there is no irreversibility constraint, the firm would always invest(disinvest) until the marginal value of capital equals to the price of capital. Since the investment adjustment cost is linear in the amount of capital invested, capital stock adjustment takes place immediately.[7] However, the irreversibility constraint here prevents the firm from adjusting its capital stock down if the current level of capital is larger than the optimal level of capital stock. Therefore, the investment policy in this model is

[7] If the capital adjustment features quadratic adjustment cost, the capital stock adjustment process takes time to finish.

characterized by the investment threshold $b(Z)$. At the threshold, the marginal value of capital equals to the price of capital. A firm expands its capital stock immediately to the threshold if its capital stock is lower than that. But the firm cannot downsize the capital stock if it is higher than the threshold.

Figure 2.7 graphs the investment threshold against firm productivity for two levels of idiosyncratic volatility. Two observations can be made. An increase of idiosyncratic productivity volatility has significant negative effect on the level of investment threshold. The investment threshold is significantly lower when the idiosyncratic volatility is high, which means the optimal investment policy allows the capital stock to fall farther before triggering positive investment. Another way to put it is firms would invest to reach a lower capital stock level when there are investment opportunities. In my baseline calculation, the increase in idiosyncratic productivity shock volatility from $\sigma = 0.1$ to $\sigma = 0.2$ lead to about 40 percent decrease in aggregate investment and about 45 percent fall in the long-run level of aggregate capital stock in steady states. The quantitative implications highlight the importance of volatility in determining capital investment and the accumulation of capital.

Second, the investment threshold function over productivity flattens when the idiosyncratic volatility is higher. This is consistent with the notion that firms are more cautious in undertaking investment projects when the volatility is higher. This finding is similar to results reported by Bloom (2009) and others.

Figure 2.8 displays the surface of firm value for different levels of productivity and capital stock. The firm's value function is increasing with respect to capital stock size and the level of productivity. These are straightforward implications from the model. The ridge on the surface represents that the value function has kinks when the irreversibility constraint just starts to bind.

2.5.4 The Transition Dynamics

In this section, I analyze the transition dynamics of the model from the low volatility steady state to the high volatility one. When the idiosyncratic volatility of productivity shocks changes, the transition dynamics are important to answer questions such as: how long does it take for firms to reallocate capital to the new long-run level and how important are these changes to short-run fluctuations?

Figure 2.9 shows the transition dynamics of consumption, investment, interest rates and the level of capital stock in response to an increase in the idiosyncratic productivity shock volatility from $\sigma = 0.1$ to $\sigma = 0.2$. Investment immediately falls as the idiosyncratic productivity shock volatility goes up. The effect is due to the real option channel. When investment is irreversible, the real option value of waiting increases as the productivity shock volatility shoots up. The fall in investment has a negative effect on the accumulation of capital, thus slowing down the growth of the economy. This results in a lower expected growth path of consumption and lower interest rates as implied by the consumption Euler equation.

An important question to be asked is how does the intertemporal elasticity of substitution $1/\psi$ plays in quantifying the general equilibrium effect. Figure 2.10 compares the transition path of model under two specification of the ψ. Even though economists haven't reached a concensus about the elasticity of substitution, I consider two benchmark values $\psi = 5$ and $\psi = 0.5$. The first value is widely used by macroeconomists and the latter one is the benchmark setup in the long-run risk literature Bansal and Yaron (2004).

As shown by the graph, the intertemporal elasticity of substitution plays an important role in quantifying the general equilibrium effect. When ψ is larger, there is a stronger response to interest rates, which counteract the partial equilibrium real option effect. The larger is the ψ, the smoother is the transition dynamics of consumption, investment and capital stock.

2.5.5 Investigating the General Equilibrium Channel

The real interest rates are the medium through which the general equilibrium channel operates. The stronger is the response to interest rates, the larger is general equilibrium effect, which counteracts the partial equilibrium real option effect. Since real interest rates can be empirically measured, the general equilibrium channel may be tested through the response of real interest rates to changes in idiosyncratic volatility.

To explore the aggregate effects of the volatility of firm productivity shock, I consider a baseline regression of real interest rate, idiosyncratic volatility of the form:

$$\Delta r_t = \beta_0 + \beta_1 \Delta \sigma_{\epsilon,t} + \epsilon_{\sigma,t} \tag{2.34}$$

where r_t is the log of the time t to $t+1$ risk-free rate. The expected real interest rate is given by subtracting the predicted inflation from the log of the nominal interest rate. The nominal interest rate is measured as the annualized Treasury-bill rate from the Federal Reserve Bank of St.Louis (TB3MS) series. Annual inflation is calculated as the log of Consumer Price Index (CPI) in December in year t, divided by CPI in December of year $t-1$. This is modeled as using an ARMA(1,1) process, and the predicted value is used as the estimate of expected inflation as in Constantinides and Ghosh (2011).

I take the first difference of interest rates and the level of idiosyncratic volatility because they seem to have non-stationary components. It is found that there is a modest negative relationship between the volatility of firm level productivity shock and the real risk-free rate. The regression coefficient is -0.16 with a t-statistics of -1.6. Table 2.3 presents the summary statistics and Figure 2.1 and 2.11 plots the time-series of idiosyncratic volatility, real and nominal interest rates. I also consider using longer term interest rates, such as 10-year Treasury rate, as the proxy for interest rates. The quantitative results remain mostly similar. This is a moderate degree of negative relationship between changes in interest rates and idiosyncratic volatility.

The next exercise I consider is to compute the model implied impulse responses of interest rates to changes in idiosyncratic volatility. I calculate the elasticity of intertemporal substitution to match the model implied impulse response to the empirical response of interest rates. That inverse intertemporal elasticity of substitution is computed is to be 1.4, which suggests that the general equilibrium effect is modest but no big enough to largely offset the partial equilibrium effect. In terms of the response of consumption, investment, the general equilibrium effect tamer the partial equilibrium effect by about 20%.

2.6 Conclusion

This paper documents robust evidence on the upward trend in idiosyncratic volatility of productivity shocks. While the volatility of firm cash flow growth, firm stock return are documented to display such upward trend, the increase in the volatility of productivity shocks may be the fundamental reason behind the rise in firm-level risk. This finding contributes to the literature on the firm-level risk.

To quantitatively investigate the consequences of the upward trend in idiosyncratic productivity shock volatility, I build a dynamic general equilibrium model with firm heterogeneity. The increase in idiosyncratic volatility has a significant negative effect on the firm investment and capital accumulation both at the short-run and long-run.

The doubling of idiosyncratic productivity shocks volatility could lead to about 40% decrease in long-run aggregate investment and capital stock.

The short-run effects on investment and capital expenditure work through two channels. The first effect works through is the partial equilibrium real option effect. When the volatility of productivity shocks is high, the real option value of waiting increases and firms thus delay their investments. The second channel works through the general equilibrium effect of interest rates on investment. In equilibrium, the fall in aggregate investment corresponds to expected future decline in consumption growth and thus lower real interest rates. The decrease in interest rates would spur investment and thus counteract the partial equilibrium real option effect. Under different parameters for the intertemproal elasticity of substitution, the size of general equilibrium effect varies. The smaller is the intertemproal elasticity of substitution, the larger is the counteracting general equilibrium effect, the slower is the speed of transition to new steady states. Pinning down the magnitude of the general equilibrium effect remains an important open question and deserves further research.

2.7 Tables and Figures

This table reports the parameter values used in the calibration of the model. The table is divided into two categories: preferences and technology.

Table 2.1: The Dynamics of Idiosyncratic Volatility

Variables	σ_{ϵ_t}	Variables	$\Delta\sigma_{\epsilon_t}$
$\sigma_{\epsilon_{t-1}}$	0.91	$\Delta\sigma_{\epsilon_{t-1}}$	-0.002
	(0.05)		(0.15)

The table reports the annual AR(1) coefficients for the level the change of idiosyncratic volatility. Standard errors are reported in parentheses. The data is from Compustat database.

Table 2.2: Calibration

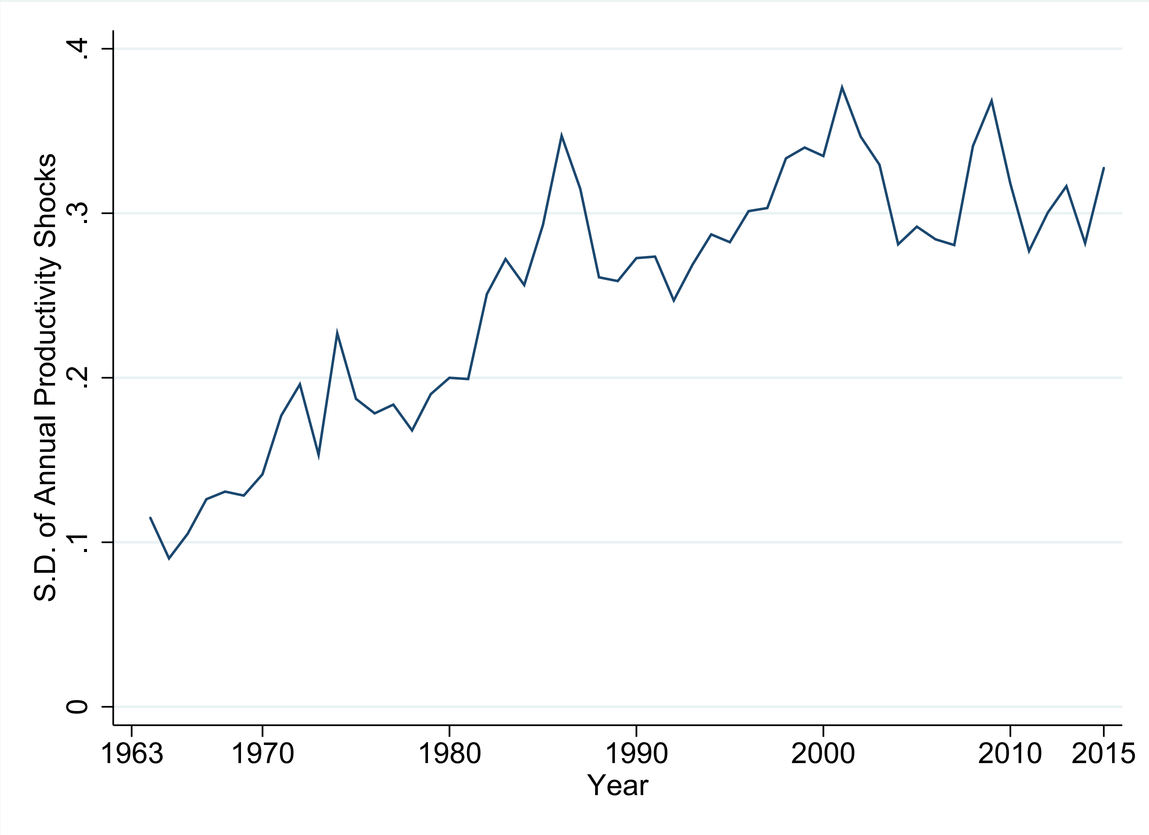
Parameter	Description	Value
Panel A: Preferences		
ρ	Time Preference Rate	0.05
φ	Inverse of Intertemporal Elasticity of Substitution	0.3
Panel B: Technology		
α	Capital share	0.33
δ	Depreciation rate of capital stock	0.10

Table 2.3: The Impact of Idiosyncratic Volatility on the Risk-free Interest Rate

	1963-2015
Variables	$\Delta\sigma_{\epsilon_t}$
Δr_{3m}	-0.16
	(0.10)
Observations	51
t-statistic	-1.6

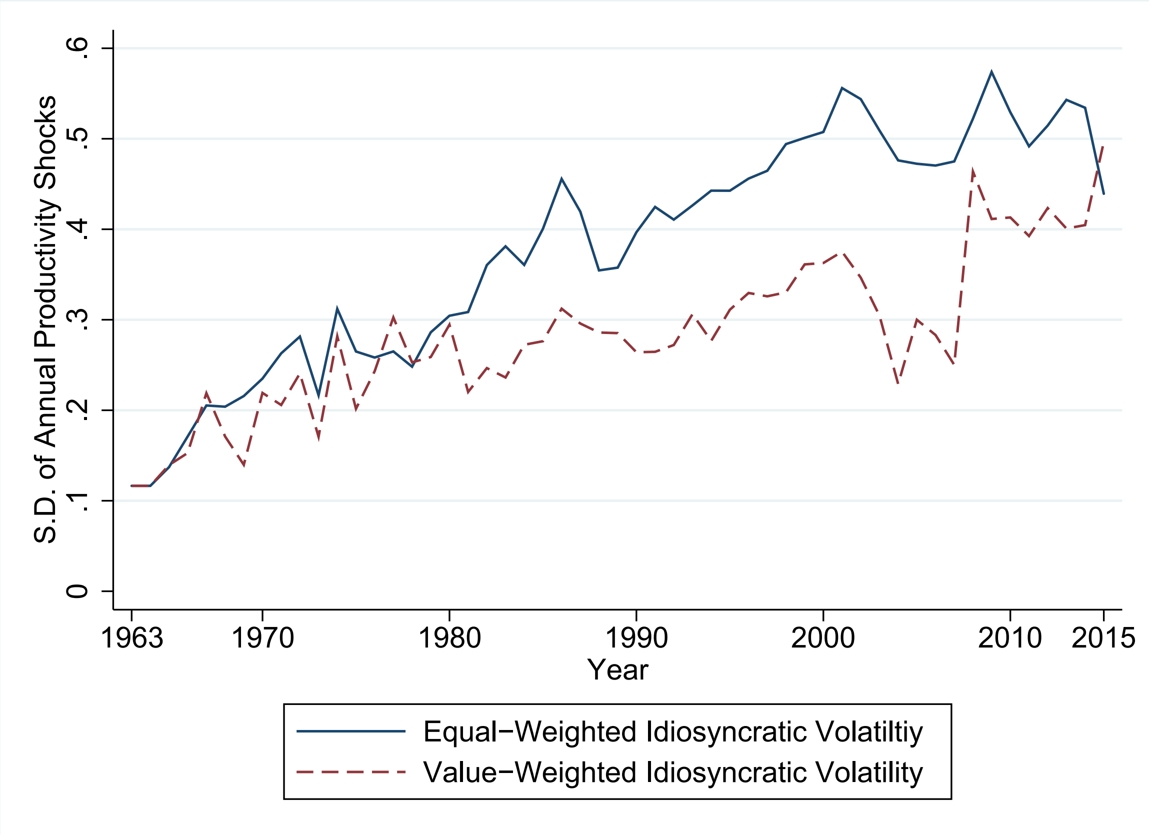
This table reports the coefficient of regressing the change of real interest rates on the change of idiosyncratic volatility. The sample period is from 1963 to 2015 annually. The data on interest rates is from the Federal Reserve Bank of St. Louis, and the idiosyncratic measure is constructed using the Compustat database.

Figure 2.1: The Idiosyncratic Volatility of Productivity Shocks



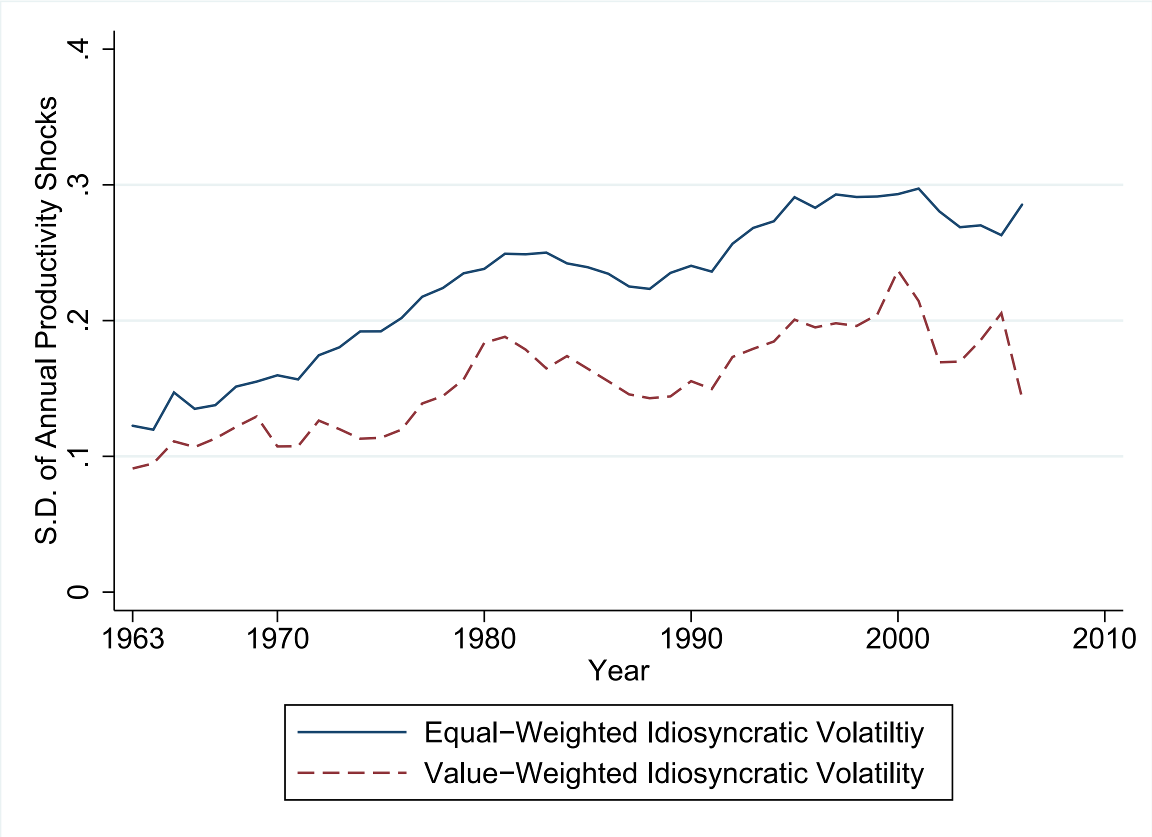
This figure plots the time-series of annual idiosyncratic productivity shocks volatility constructed by using the Compustat dataset. The sample spans from 1963 to 2015. The volatility is defined as the cross-sectional dispersion of annual productivity shocks across firms.

Figure 2.2: The Idiosyncratic Volatility of Productivity Shocks Using a Time-Series Method



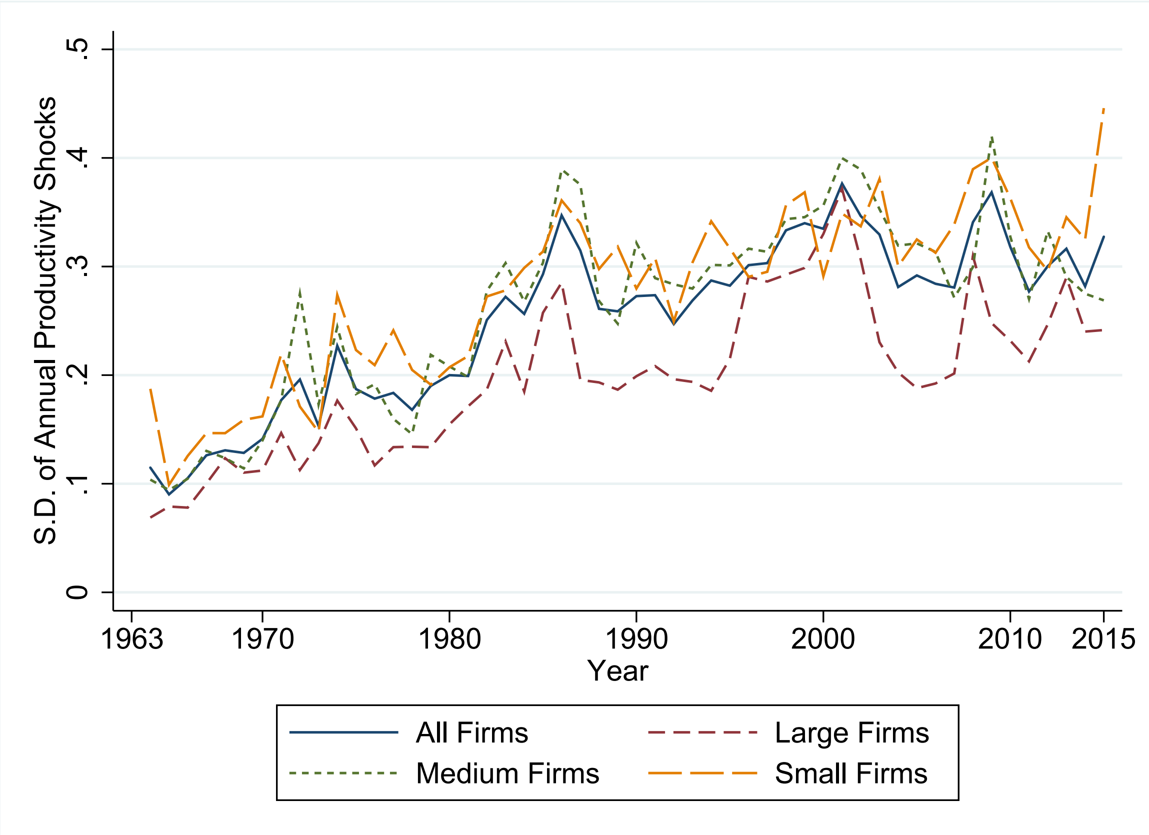
This figure plots the time-series of annual idiosyncratic productivity shocks volatility by using the method in section 2.2.3. I use the Compustat and CRSP datasets from 1963 to 2015. The dotted line gives the value-weighted idiosyncratic volatility measure, which weights changes of firm productivity by firm market equity values. The solid line plots the equal-weighted measure, which doesn't take firm market equity values into consideration.

Figure 2.3: The Idiosyncratic Volatility of Productivity Shocks Using Rolling S.D.



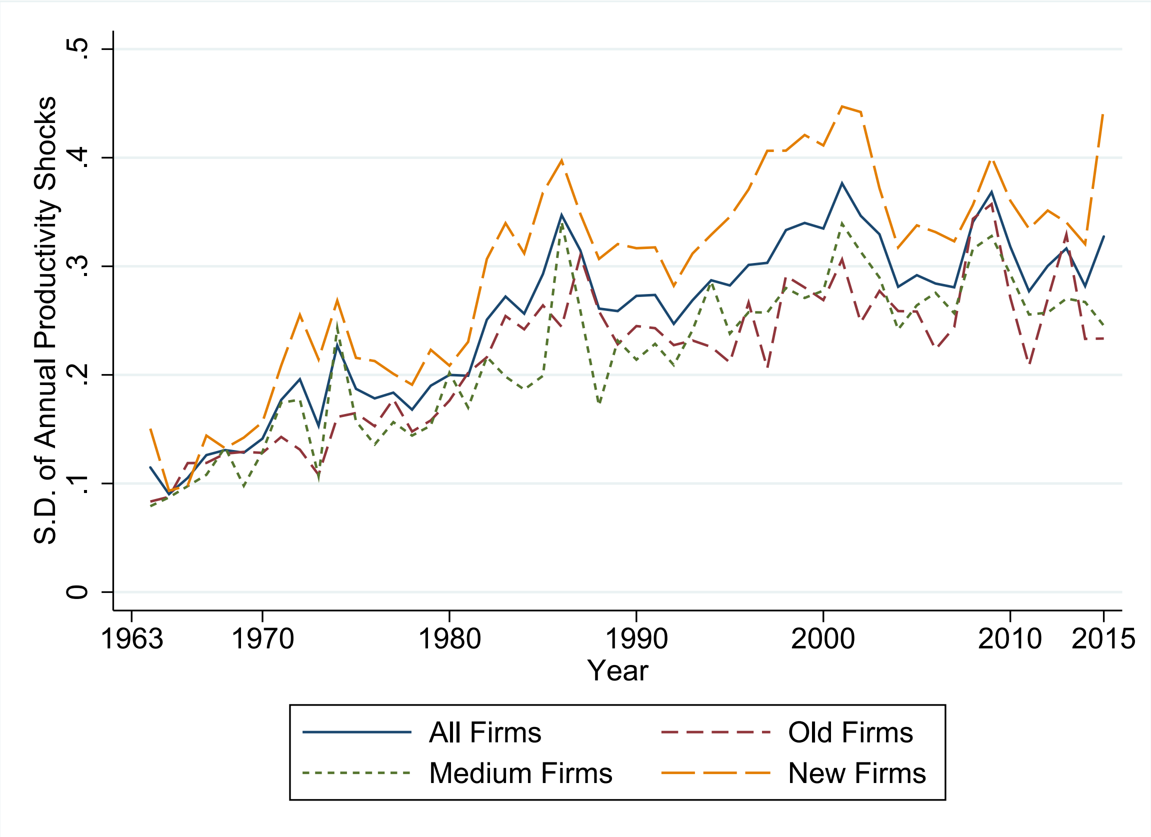
This figures the time series of annual idiosyncratic productivity shocks volatility by using rolling-window standard deviations method in section 2.2.4. The data spans from 1963 to 2015 using Compustat database.

Figure 2.4: The Idiosyncratic Volatility of Productivity Shocks Controlling for Size



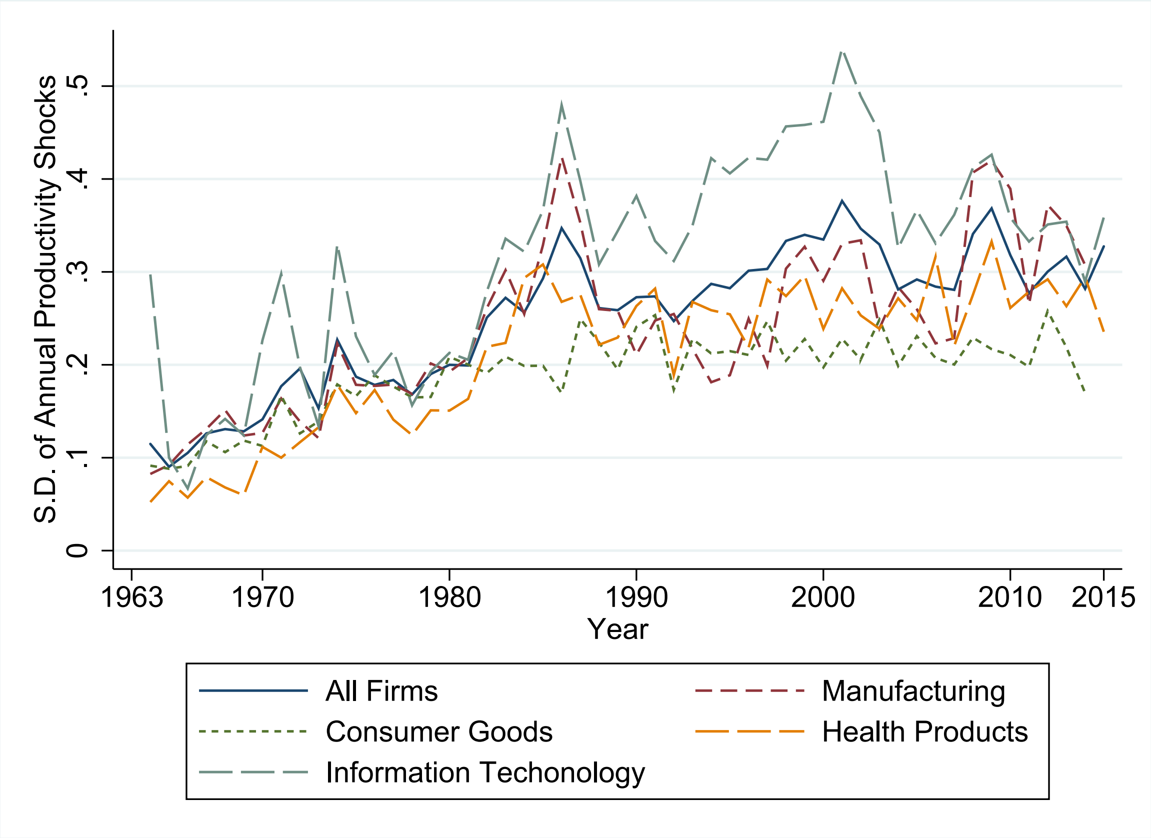
This figure plot the time series of the annual idiosyncratic productivity shocks volatility for firms in different size groups. The idiosyncratic volatility is measured as the cross-sectional dispersion of annual productivity shocks. The sample spans from 1963 to 2015. I use market equity values to measures firm sizes.

Figure 2.5: The Idiosyncratic Volatility of Productivity Shocks Controlling for Age



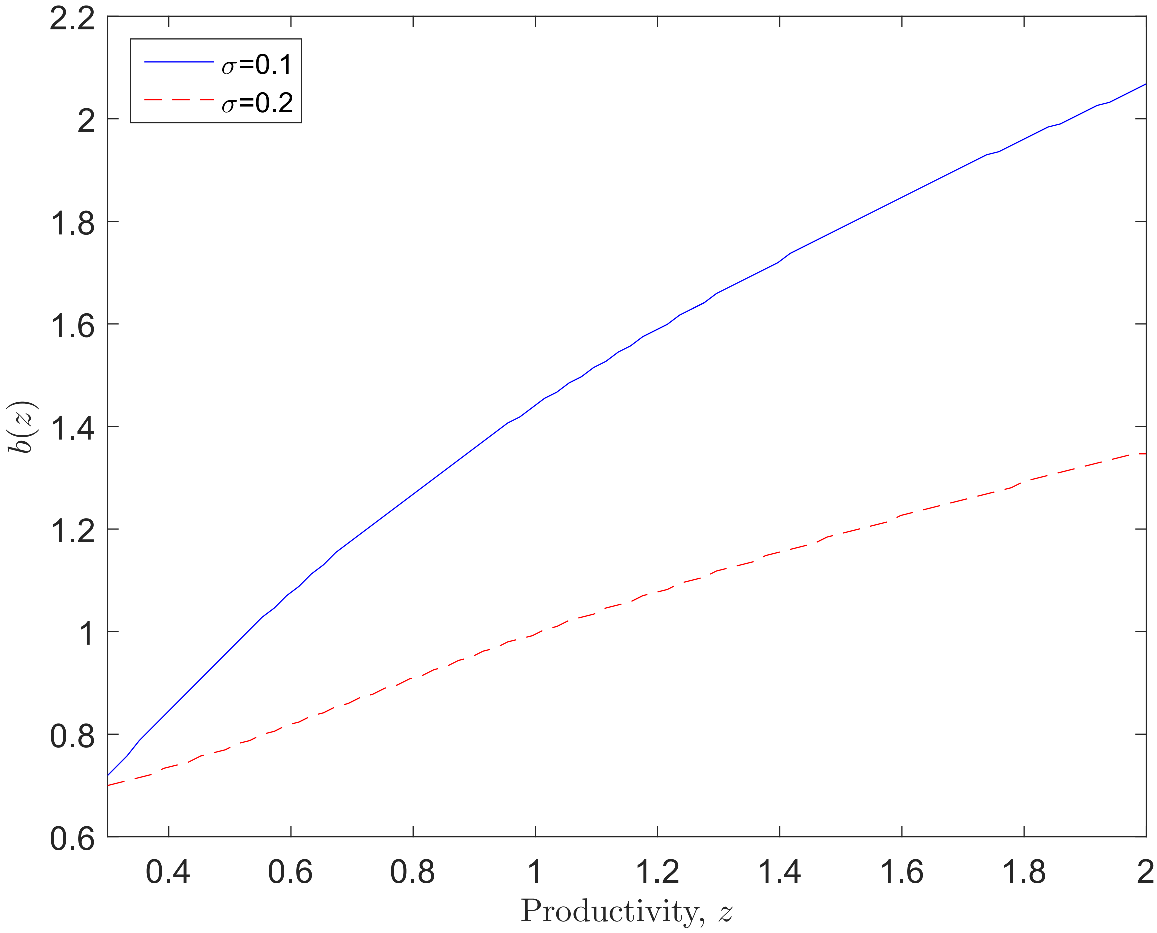
This figures the time series for the annual idiosyncratic productivity shocks volatility for different age groups. The idiosyncratic volatility is measured as the cross-sectional dispersion of annual productivity shocks. The sample spans from 1963 to 2015.

Figure 2.6: The Idiosyncratic Volatility of Productivity Shocks Controlling for Sectors



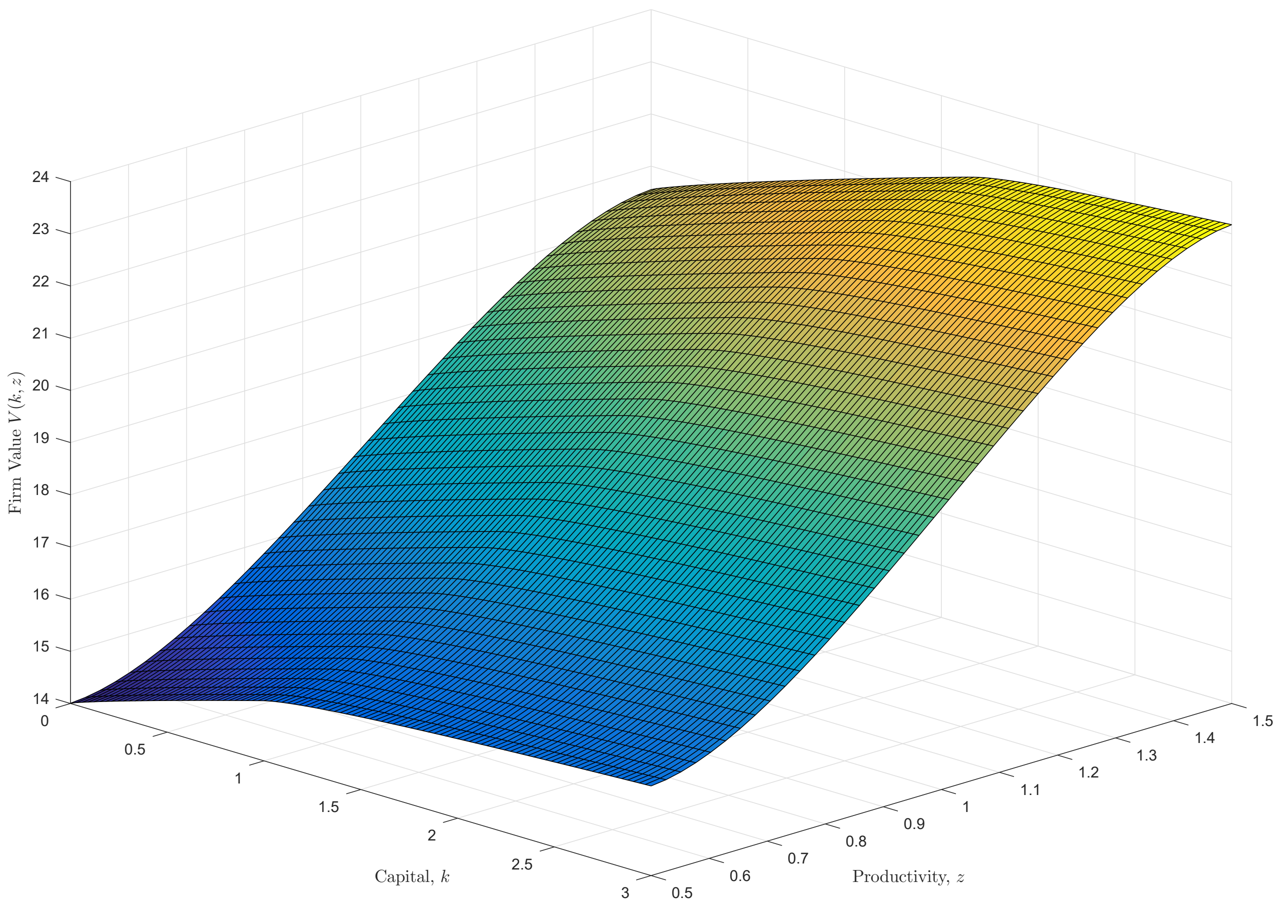
This figure plots the time series for the annual idiosyncratic productivity shocks volatility for different sectors. The idiosyncratic volatility is measured as the cross-sectional dispersion of annual productivity shocks. The sample spans from 1963 to 2015. I consider manufacturing, consumer goods, health products and information technology sectors. Detailed definitions of sectors can be seen in the main text.

Figure 2.7: The Investment Threshold



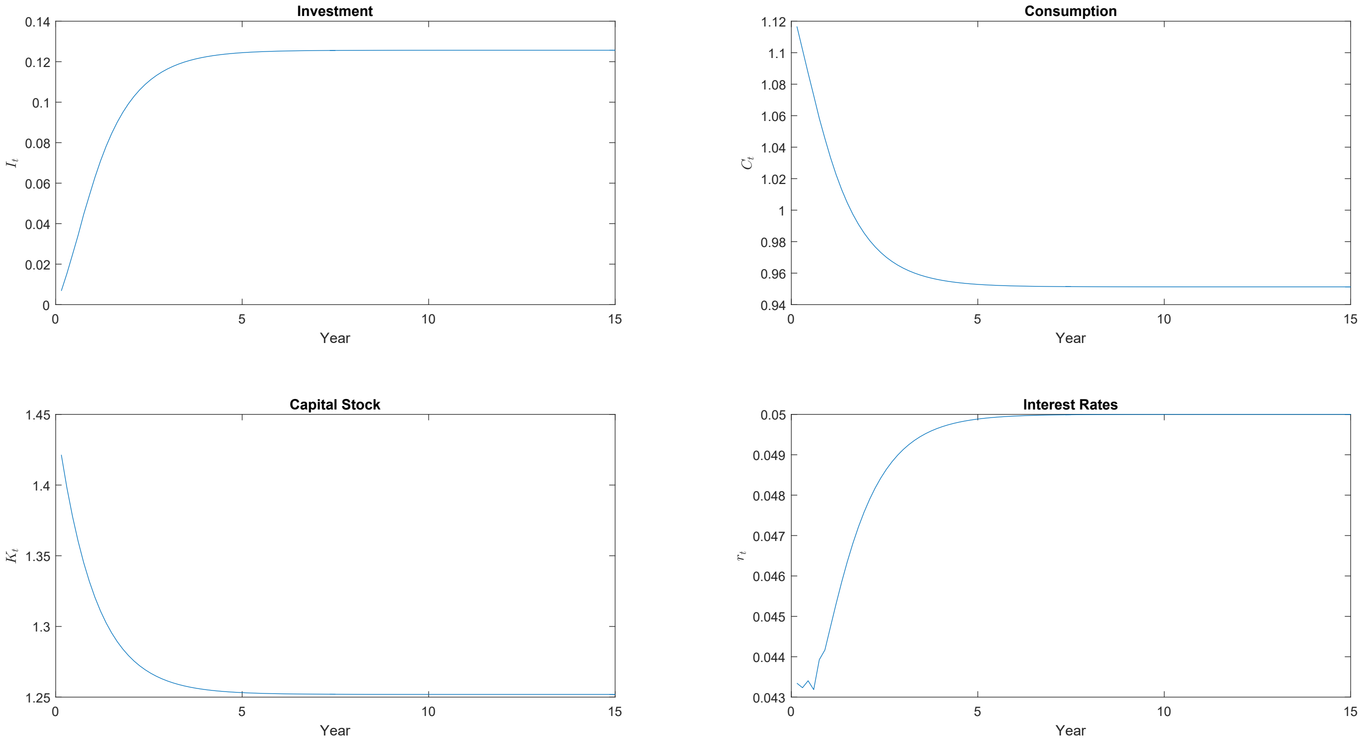
This figure plots the investment threshold function $b(z)$ given parameters in Table 2.2. The volatility of productivity shocks σ equals 0.1 and the inverse of the intertemporal elasticity of substitution equals 0.5.

Figure 2.8: The Value of Firms over Productivity and Capital



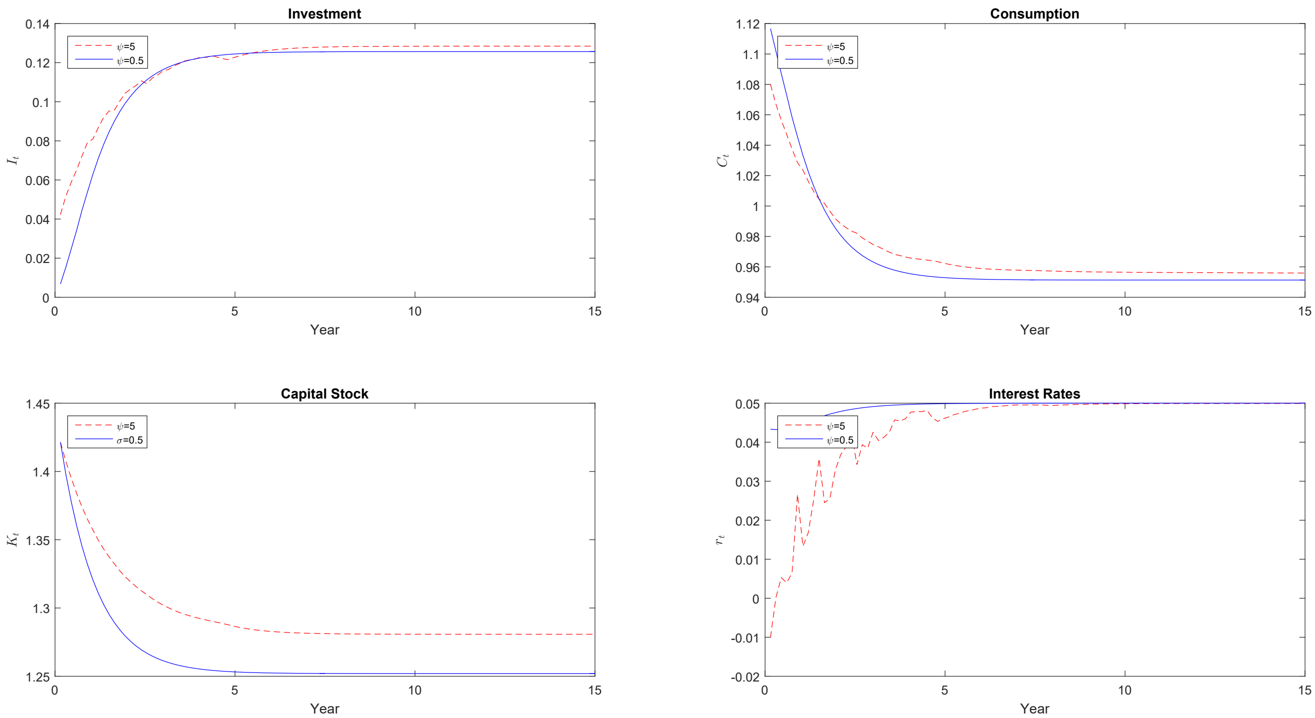
This figure plots the value of firms over productivity and capital stock computed using the numerical finite difference method. The ridge on the surface represents the place when the irreversiblity constraint starts to bind. The parameters of the model are listed in Table 2.2. The volatility of productivity shocks σ equals 0.1 and the inverse of the intertemporal elasticity of substitution equals 0.5.

Figure 2.9: The Transition Dynamics of Consumption, Investment, Capital Stock Level and Interest Rates



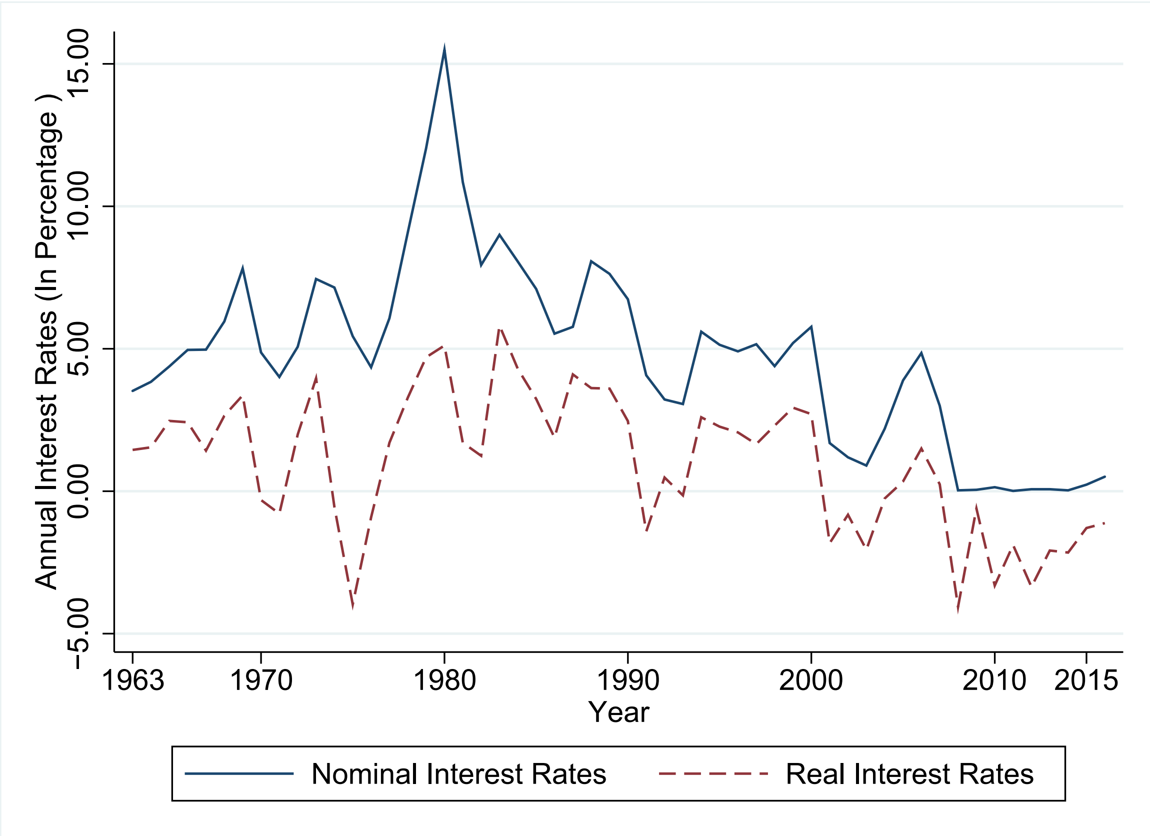
The figures plots the transition dynamics of investment, consumption, capital stock and interest rates to a unexpected permanent change of $\sigma = 0.1$ to $\sigma = 0.2$. The units on the horizon axis represent the time after which the change happens measured in years. The inverse of the intertemporal elasticity of substitution ψ equals to 0.5.

Figure 2.10: The Transition Dynamics and Elasticity of Intertemporal Substitution



The figures plots the transition dynamics of investment, consumption, capital stock and interest rates to a unexpected permanent change of $\sigma = 0.1$ to $\sigma = 0.2$. The units on the horizon axis represent the time after which the change happens measured in years. The inverse of the intertemporal elasticity of substitution ψ equals to 0.5 and 5.

Figure 2.11: The Time Series of Nominal and Real Interest Rates



This figures plot the time-series for the nominal and real interest rates, which are measured in percentage points. The data used to construct this figure is from Federal Reserve Bank of St.Louis. The sample spans from 1963 to 2015 annually. The nominal interest rate is the annualized 3-month Treasury Bill rate. And the real rate is defined as the nominal interest minus expected inflation, which is calculated using the method in section 2.5.5.

Chapter 3

Pricing the Short-Run and Long-Run Components of Idiosyncratic Volatility

3.1 Introduction

Whether a stock's expected return depends on idiosyncratic risk has been a central question in the asset pricing literature. Classical theory such as Capital Asset Pricing Model (CAPM, Sharpe (1964)) suggests that idiosyncratic risk should not be priced because they can be diversified away. Our standard theories build upon the assumptions of rational investors, complete diversification, and no trading frictions. But investors, in reality, may not hold perfectly diversified portfolios. Earlier studies such as Merton (1987), argue that idiosyncratic risk should be rewarded with higher expected returns if investors cannot diversify their portfolios properly.

Recent empirical studies on this topic instead tend to find a negative relation between expected return and idiosyncratic volatility. The papers by Ang et al. (2006) and Ang et al. (2009b) show that stocks with high idiosyncratic risk, defined as the standard deviation of the residuals from the Fama and French (1993) (hereafter FF) model estimated with daily returns in the previous month, have anomalously low returns in the subsequent month. This finding is challenging for theories that suggest that idiosyncratic volatility should be irrelevant or positively related to expected returns.

For rational investors, the idiosyncratic volatility measure that matters for them is arguably the conditional idiosyncratic volatility. Since idiosyncratic volatility is strongly time-varying, the realized idiosyncratic volatility in the last month may not a good proxy for the conditional idiosyncratic volatility. Fu (2009) uses exponential generalized autoregressive conditional heteroskedasticity (EGARCH) models to estimate conditional idiosyncratic volatility and find that a significantly positive relationship between the estimated conditional idiosyncratic volatility and expected returns. Chua et al. (2010) model idiosyncratic volatility as a second-order autocorrelation process and also finds that the relationship between expected return and conditional idiosyncratic volatility is positive.

However, existing models for estimating conditional idiosyncratic volatilities don't well capture the dynamics of idiosyncratic volatility in the long-run. I document empirical evidence that idiosyncratic volatility persists for long periods of time: more than a year. But it has been noted in the literature (Andersen et al. (2003), for example) that generalized autoregressive conditional heteroskedasticity (GARCH) and similar models don't well capture the persistence of stock return volatility in the long run. Thus, those models may produce biased proxies for the conditional idiosyncratic volatility. Corsi (2009) proposes an additive cascade model of volatility defined over different time frequencies to capture the persistence of aggregate stock return volatility. To parsimoniously capture the dynamics of idiosyncratic volatility, I build a model for idiosyncratic volatility featuring a short-run component and more persistent long-run component.

Another reason why the long-run component is important for investors may be trading costs, which prevent them from actively adjusting portfolios. Thus, these investors might care about the conditional volatility of a stock's return over a horizon longer than a month. In this case, the long-run component is an even better measure of the conditional volatility. Yet another possibility is that the conditional volatility of a stock's idiosyncratic return is proxying for a common risk factor. If this is the case, we don't have a theory that says whether the common risk factor is the short-run component of volatility or the long-run component. It is worth investigating which component it might be.

This paper presents a new model for idiosyncratic volatility, in which the idiosyncratic volatility of stock returns consists of two components. One of these components is long-run, and it can be modeled as (fully) persistent. The other component is short-run and has mean zero. This approach parsimoniously captures shocks to volatilities at different horizons. I estimate the model for idiosyncratic volatility and decompose the volatility into short-run and long-run components. I find that there is a significant negative relationship between expected long-run volatility and expected return. However, I don't find any significant relationship between expected short-run volatility and expected return. These findings are in sharp contrast to Fu (2009), Huang et al. (2010) and other papers that construct measures of expected idiosyncratic volatilities. While these papers tend to find a positive relationship between expected idiosyncratic volatility and expected return, my results lend strong support to a negative relationship between expected long-run volatility and expected return. The crucial difference between my paper and theirs is the ability to capture the persistence of idiosyncratic volatility in the long-run, which has strong asset pricing implications.

I study two types of models for idiosyncratic volatility. In one type of the model, the long-run component is constrained to have a unit root and the short-run component is a white noise. The other type of the model doesn't impose these restrictions. Parameter estimates from the model using asset prices data suggest that these restrictions could be plausible assumptions about the idiosyncratic volatility process. The quantitative asset pricing implications are also similar. There is a significantly negative relationship between expected long-run volatility and expected returns.

There have been several papers in the literature that demonstrate short-run and long-run components of volatility have different asset pricing implications. Lee and Engle (1993) show that aggregate stock market volatility is subject to shocks at different frequencies. Adrian and Rosenberg (2008) find that the short-run and long-run components of equity market volatility are differently priced risks and are important cross-sectional stock returns factors. Christoffersen et al. (2008) propose that short-run and long-run components of volatility have important implications for option pricing. In this paper, I decompose idiosyncratic volatilities at the firm level into short-run and long-run components and ask whether exposures to short-run and long-run volatilities have different asset pricing implications.

3.2 Estimating Idiosyncratic Volatilities

This section describes the data and methods used to estimate idiosyncratic volatilities.

3.2.1 Data

I use all nonfinancial firms in the intersection of (a) the NYSE, AMEX, and NASDAQ return files from the Center for Research in Security Prices (CRSP) and (b) the merged COMPUSTAT annual industrial files of income-statement and balance-sheet data, also maintained by CRSP. I exclude financial firms with SIC code between

6000 to 6999 and utility firms with SIC code between 4900 to 4999. The CRSP returns cover NYSE and AMEX stocks until 1973 when NASDAQ returns also come on line. The COMPUSTAT data are for 1962-2015. The 1962 start date reflects the fact that book value of common equity (COMPUSTAT item 60), is not generally available prior to 1962. More importantly, COMPUSTAT data from earlier years have a serious selection bias; the pre-1962 data are tilted toward big historically successful firms.

The procedures below are standard in the literature following Fama and French (1992). To ensure that the accounting variables are known before the returns they are used to explain, I match the accounting data for all fiscal year ends in calender year $t-1$ with the returns for July of t to June of $t+1$. The 6-month (minimum) gap between fiscal year end and the return tests is conservative. I use a firm's market equity at the end of December of year $t-1$ to compute its book-to-market ratio for $t-1$, and I use its market equity for June of year t to measure its size. Thus, to be included in the returns tests for July of year t, a firm must have a CRSP stock price for December of year $t-1$ and June of year t. It must also have monthly returns for at least 24 of the 60 months preceding July of year t. And the firm must have COMPUSTAT data on total book assets, book equity, and earnings, for its fiscal year ending in (any month of) calender year $t-1$. [1]

[1] Fiscal years are referenced by their end date or end year.

3.2.2 Idiosyncratic Volatility Measure

Following Ang et al. (2006) and Bali and Cakici (2008), I concentrate on idiosyncratic volatility measured relative to the FF-3 model. Specifically, for each month, I run the following three-factor regression for each firm,

$$r^i_{t,d} = \alpha^i_t + \beta^i_{MKT} MKT_{t,d} + \beta^i_{SMB} SMB_{t,d} + \beta^i_{HML} HML_{t,d} + \epsilon^i_{t,d} \tag{3.1}$$

where for day d in month t, $r^i_{t,d}$ is stock i's excess return, $MKT_{t,d}$ is the market excess returns, $SMB_{t,d}$ and $HML_{t,d}$ capture size and book-to-market effects, respectively. The residuals $\epsilon^i_{t,d}$ is the idiosyncratic risk for month t. I define the idiosyncratic volatility of stock returns for firm i v^i_t as

$$v^i_t = \sqrt{var(\epsilon^i_{t,d}) N_m} \tag{3.2}$$

where N_m is the number of trading days in month t for firm i. It is useful to note that the idiosyncratic volatility v^i_t is the daily standard deviation of residuals times the square root of the number trading days in that month. The inclusion of N_m transforms the daily return residuals into monthly residuals. This procedure can be can be seen at French et al. (1987) and Fu (2009). Since the conditional volatility v^i_t cannot be directly observed, I use the squared daily return residuals in month t to measure the individual stock's idiosyncratic volatility for month t

$$IV_t^i \equiv \sqrt{\sum_{d=1}^{N_m} (\epsilon_{t,d}^i)^2} \tag{3.3}$$

When I refer to idiosyncratic volatility in this paper, I mean idiosyncratic volatility relative to the FF-3 model. To improve the precision on idiosyncratic volatility measures, I require that firms should have at least 15 trading days in a month for which the CRSP reports a daily return. Moreover, I require that firms must have at least 30 months of estimates for the idiosyncratic volatility to be used for my empirical analysis.

3.2.3 Time Series Properties of Realized Idiosyncratic Volatility

Table 3.1 presents the time-series properties of the realized idiosyncratic volatility (IV). I first compute the time-series statistics of idiosyncratic volatility for each firm and then summarize the mean statistics across about 22,000 firms. The mean of idiosyncratic volatility is on average 16.77% across stocks and the mean standard deviation for IV is 9.85%. The skewness is 1.84 and kurtosis is 7.22 which suggests that the idiosyncratic volatility is positively skewed and fat-tailed. The autocorrelation for the realized idiosyncratic volatility is 0.33 at lag 1, 0.26 at lag 2, 0.16 at lag 5, 0.10 at lag 10, and 0.09 at lag 12. The autocorrelation of 0.33 at one month lag and 0.26 at two months lag suggests that shocks to idiosyncratic volatility are not very persistent at the short horizon (within a quarter). However, the autocorrelations decay slowly over longer periods, for example over a year after shocks. The autocorrelation of realized idiosyncratic with a lag period of 12 months is still almost 0.1. In comparison, the autocorrelation of an AR(1) process with first order autocorrelation of 0.33 would predict that the autocorrelation with 12 months lag is less than 0.02 basis point (One basis point equals to one hundredth of one percentage point).

This pattern suggests that idiosyncratic volatility displays long-memory properties: persistence of the volatility that lasts for long time periods (years). Corsi (2009) proposes that the long memory property of aggregate stock return volatility could be captured by a additive cascade model of volatility defined over different time periods. The empirical evidence in the section suggests that the idiosyncratic volatility of stock returns also displays long-memory property. Therefore, the realized idiosyncratic volatility could thus be better captured by a process with components of different persistences. I model the log of idiosyncratic volatility as the sum of a short-run and long-run component. The short-lived component is less persistent and has a large impact on the autocorrelations of idiosyncratic volatility at short horizons (over a quarter). There also exists a persistent long-run component, which dominates the autocorrelations at longer horizons (over a year and further).

3.2.4 Decomposing Idiosyncratic Volatility

To decompose the idiosyncratic volatility into short-run and long-run components. I model the idiosyncratic volatility v_t^i follows

$$\begin{aligned}\textbf{Idiosyncratic Volatility} &: \log v_t^i = s_t^i + l_t^i \\ \textbf{Short-run Component} &: s_{t+1}^i = \rho_s^i s_t^i + \sigma_s^i \epsilon_{s,t}^i \\ \textbf{Long-run component} &: l_{t+1}^i = \phi_i + \rho_l^i l_t^i + \sigma_l^i \epsilon_{l,t}^i\end{aligned} \tag{3.4}$$

I refer this model as the short and long-run (SL) model hereafter. In equation (3.4), the log-volatility is the sum of two components, s_t and l_t. The short-run component s_t is a mean-reverting process with mean zero and shocks to the short-run component die off quickly over time. The long-run component l_t is a persistent component. I normalize the process such that $\rho_l > \rho_s$ and σ_s and σ_l are the volatility of shocks to the short-run and long-run components.. This restriction identifies the model as otherwise the two components can be interchangable. Adrian and Rosenberg (2008) consider a two components model for aggregate stock market volatility and find that the prices of risk are different for the short-run and long-run component. My paper differs from theirs in the focus on idiosyncratic volatility of stock returns.

For each firm, equation (3.4) is readily in the state space form and the unobserved short-run and long-run components can be directly estimated via Kalman filter. I consider $\hat{s}_t \equiv \mathbb{E}_t(s_{t+1}|y_1, y_2, ...y_t)$ and $\hat{l}_t \equiv \mathbb{E}_t(l_{t+1}|y_1, y_2, ...y_t)$ as the expectation for the short-run and long-run components at time $t+1$ based on information available at time t. The smoothed estimates $\mathbb{E}(s_t|y_1, y_2, ...y_T)$ and $\mathbb{E}(l_t|y_1, y_2, ...y_T)$, which use the whole sample information, may produce more precise estimates for the unobserved components s_t and l_t at each point in time. For empirical issues, forming conditional expectations by using $\hat{s}_t$ and $\hat{l}_t$ may be preferred as the analysis in the paper applies to real-world financial markets. Portfolios sorted on the expected short-run volatility $\hat{s}_t$ and long-run volatility $\hat{l}_t$ could be implemented as trading strategies. Hence, in this paper, I only report summary statistics for portfolios formed by using the current information to the market. Summary statistics using the smoothed estimates are available upon request. Portfolios performance sorted by filtered and smoothed estimates are still largely similar.

3.2.5 A Permanent and Transitory Special Case

In my empirical work, I also investigate a special case of equation (3.4) where the long-run component is permanent and follows a random walk and the short-run component is purely transitory and follows a white noise. I refer this model as the permanent and transitory (PT) model.

$$\textbf{Idiosyncratic Volatility} : \log v_t^i = s_t^i + l_t^i \tag{3.5}$$
$$\textbf{Short-run Component} : s_{t+1}^i = \sigma_s^i \epsilon_{s,t}^i$$
$$\textbf{Long-run component} : l_{t+1}^i = l_t^i + \sigma_l^i \epsilon_{l,t}^i$$

Equation (3.5) may be viewed as a special case of (3.4) with the restriction that $\rho_s = 0$ and $\rho_l = 1$. The log-volatility is the sum of two components, s_t and l_t. The short-run component s_t is a white noise process and the long-run component l_t follows a random walk. Thus, changes to the long-run volatility could be permanent and are persistent over time. For each firm, equation (3.5) can also be estimated using Kalman filter. Since the long-run component l_t follows a random walk, the one-step-ahead conditional expectation of the long-run component $\mathbb{E}_t(l_{t+1}|y_1, y_2, ..., y_t) = \mathbb{E}_t(l_t|y_1, y_2, ..., y_{t-1})$ and that of the short-run component $\mathbb{E}_t(s_{t+1}|y_1, y_2, ..., y_{t-1})$ trivially equals to zero. Therefore, for the permanent and transitory (PT) model, I use the expectation for the contemporaneous short and long-run component $\tilde{s}_t = \mathbb{E}_t(s_t|y_1, y_2, ..., y_t)$ and $\tilde{l}_t = \mathbb{E}_t(l_t|y_1, y_2, ..., y_t)$ as variables on which to sort portfolios at time t. What remains to assume is the structure of v_t.

3.2.6 Parameter Estimates of the Idiosyncratic Volatility Model

In practice, the true conditional idiosyncratic volatility v_t cannot be directly observed. The realized volatility measurement IV_t is subject to measurement errors. Andersen et al. (2004) theoretically find that population forecasts of the volatility formed by projecting the volatility on the history of the realized volatilities are almost as accurate as forecasts formed by projecting on the (unattainable) history of latent volatilities for the general class of eigenfunction stochastic volatility models. This implies, among other things, that the measurement error component is largely irrelevant for forecasting. Moreover, the measurement errors are close to independently and identically distributed, which means they have little forecasting power. Since the measures of the short-run and long-run component ($\hat{s}_t$ and $\hat{l}_t$, for example) are forecasts based upon the history of idiosyncratic volatility, the existence of measurement errors may be largely unimportant for the conclusion of the paper. Therefore, I directly fit the two components model to the realized idiosyncratic volatility IV_t^i.

Table 3.2 summarizes the parameter estimates for the short-run and long-run volatility (SL) model with equation (3.4) and the permanent transitory volatility (PT) model with equation (3.5). Both the SL and PT model are estimated using the maximum likelihood method. For the SL model, the mean AR(1) parameter for the short-run component is -0.06 while the median is 0.003. The long-run component is more persistent with mean AR(1) coefficient of 0.78 and median of 0.94. The mean volatility of shocks to the short-run component is 0.11 and the median is 0.09 . For the long-run component, the mean volatility is 0.07 and the median is 0.02.

The permanent and transitory (PT) model can be viewed as a special case of the SL model with $\rho_s = 0$ and $\rho_l = 1$. Given that the median estimate of ρ_s is 0.003 and ρ_l is 0.94 from the SL model, the PT model can be a plausible model to capture the dynamics of idiosyncratic volatility. For the PT model, the only parameters to be estimated are the volatility of shocks to the short-run and long-run component. The mean volatility of shocks to the short-run component is 0.14 and the median is 0.11. As for the volatility of shocks to the long-run component, the mean is 0.05 and the median is 0.01. The magnitude of shocks is also largely similar to estimates from the SL model.

3.3 Pricing Idiosyncratic Volatility in the Cross-Section

This section considers the performance of portfolios formed by different measures of idiosyncratic volatilities and asks whether exposures to different volatilities are systematically important for expected stock returns.

3.3.1 Patterns in Average Returns for Idiosyncratic Volatility

I first consider value-weighted quintile portfolios formed every month by sorting stocks based on idiosyncratic volatility relative to the Fama and French (1993) model. Portfolios are formed every month, based on realized volatility computed using daily data over the previous month. Similar to findings of Ang et al. (2006), portfolios with high realized idiosyncratic volatility have low returns. Table 3.3 shows that average returns increases from 0.92% per month to 1.00% going from the quintile 1 (low idiosyncratic volatility stock) to quintile 3. Then portfolios returns drop tremendously going to quintile 5. The portfolio with highest idiosyncratic volatility (quintile 5) has a surprisingly low average return with 0.08% per month. The difference of returns between the highest and lowest portfolio is as large as 0.84% per month, which is statistically significant with robust t-statistic of -2.98. The FF3-alpha for quintile 5, is -1.17% per month with a robust t-statistics of -8.06. Therefore, the difference in quintile portfolio returns can not be explained by our standard asset pricing model such as Fama and French (1993). The difference in average returns in 3.3 is evidence for the negative relationship between expected return and idiosyncratic volatility.

3.3.2 Portfolios Sorted by Short-Run and Long-Run Volatilities

Estimation of the state-space model of (3.4) and (3.5) produces estimates for the expected short-run volatilities and long-run volatilities at the monthly frequency. I form value-weighted quintile portfolios sorting by the short-run volatility $\hat{s}_t$ or $\tilde{s}_t$ and long-run volatility $\hat{l}_t$ or $\tilde{l}_t$. Fu (2009) argues that the use of lagged realized idiosyncratic volatilities measures by Ang et al. (2006) is not an appropriate proxy for the expected idiosyncratic volatility. My empirical analysis is based on expected short-run and long-run volatilities and are therefore free from this critique. [2]

The portfolio returns displayed in Table 3.4 and 3.5 are intriguing. Exposure to long-run volatilities is a robustly negative priced risk. For both the SL model and PT model, the portfolio with highest long-run volatilities earns very low average returns. The average return for quintile 5 portfolio with the highest long-run idiosyncratic volatility is as low as 0.35% per month for the SL model and even lower as 0.13% for the PT model. The difference in the average return between the portfolio with the lowest and highest long-run volatility is -0.52% per month for the SL model and -0.79% for the PT model. The difference in returns can also not be explained by the difference in the Fama-French model. The highest long-run volatility portfolio has FF3-alpha of -1.03% for the SL model and -1.17% for the PT model. The alphas

[2] The only exception is for $\tilde{s}_t$. The PT model has a degenerate expectation of the short-run component as a constant zero.

are statistically significant with robust t-statistics of -6.38 and -6.30. The negative relationship between expected long-run idiosyncratic volatility and expected return is slightly stronger in the PT model. This might be due to the more precision in extracting the unobserved long-run components from the absence of sampling errors for estimating ρ_s and ρ_l.

However, I don't find a significant relation between exposure to the short-run component and expected return. For the SL model, portfolios sorted by $\hat{s}_t$ or $\tilde{s}_t$ don't reveal any robust return pattern. There is little spread in average returns across portfolios. For the SL model, the quintile 1 portfolio earns a return of 0.90% per month and the quintile 5 portfolio earns 0.93%. For the PT model, the average return is 0.94% for the quintile 1 portfolio and 0.80% for the quintile 5 portfolio.

3.4 Cross-Sectional Regressions

In this section, I investigate the cross-sectional relationship between average s-tock returns and the estimated conditional idiosyncratic volatilities. I employ Fama and MacBeth (1973) regressions of the cross-section of stock returns on idiosyncrat-ic volatilities and other firms characteristics on a monthly basis and calculate the time-series averages of the coefficients. My goal is to test whether the coefficient on idiosyncratic volatility is significantly different from zero in explaining cross-sectional stock returns.

Specifically, I run the following cross-sectional regressions each month for the SL and PT model:

$$R_{i,t+1} = X_{i,t}\beta_t + \gamma_{l,t}\hat{l}_t + \gamma_{s,t}\hat{s}_t + \epsilon_{i,t+1} \tag{3.6}$$

$$R_{i,t+1} = X_{i,t}\beta_t + \gamma_{l,t}\tilde{l}_t + \gamma_{s,t}\tilde{s}_t + \epsilon_{i,t+1} \tag{3.7}$$

where $X_{i,t} = [Beta_{i,t}, Ln(Size)_{i,t}, Ln(BE/ME)]$, $Beta_{i,t}$ is the estimate for stock i's market beta in month t. The term $Ln(Size)_{i,t}$ is the log of stock i's market capitalization at the end of month t, and $Ln(BE/ME)_{i,t}$ is the log of stock i's book-to-market ratio as of the end of month t based upon the last fiscal year's information.

To obtain estimates for $Beta_{i,t}$, the procedures are to follow Fama and French (1992). I divide all stocks traded in NYSE, AMEX, and NASDAQ into ten groups by their market capitalization. Within each size decile, stocks are sorted again by their pre-ranking betas into ten groups. The pre-ranking betas are estimated on previous 60 months returns for each firm. The one hundred portfolios are then rebalanced every month. For each portfolio, I estimate portfolio betas again using the full sample of returns on each of the 100 portfolios. The full-period beta estimates of betas are assigned to each stock in the 100 portfolios. Note that assigning the full-period portfolio betas to stocks does not mean that a stock's beta is not changing over time. When the portfolio is rebalanced every month, a stock can move across portfolios with changes in the market capitalization and in the beta estimates for the preceding

5 years.

Table 3.7 shows time-series averages of the coefficients from the month-by-month Fama-Macbeth (FM) regressions of the cross-section of stock returns on size, *Beta*, and book-to-market ratio and different measures of idiosyncratic volatility. The average coefficient on variables used to explain expected returns provides standard FM tests for determining which variables on average have explanatory power during the July 1963 to December 2015 period.

The regressions in Table 3.7 say that size and book-to-market are useful variables to explain the cross-section of stock returns. When regressing on size, *Beta* and book-to-market ratio, the average coefficient from the monthly regressions on size is -0.14 with a t-statistic of -3.85. The coefficient on book-to-market ratio is 0.10 with a t-statistic of 2.74. These significant regressions persist no matter what idiosyncratic volatility measures are put into the regression. In contrast to the power of size and book-to-market to explain average stock returns, market beta is unimportant to explain the cross-section of stock returns. When size and book-to-market are controlled for, the average coefficient for *Beta* is only 0.02 with a t-statistic of 0.06.

Given that that size and book-to-market are useful benchmark variables to explain stock returns, the next step is to consider whether idiosyncratic volatility matters for the cross-section of stock returns. The average coefficient on realized idiosyncratic volatility (IV) is -0.15 with a t-statistic of -5.12. The finding confirms a negative relationship between idiosyncratic volatility and expected return.

The next important regression is to put the measure of expected short-run and long-run idiosyncratic volatility into Fama-Macbeth regressions. The regression using SL model reported in Panel A of Table 3.7 says that there exists a negative relationship between expected long-run volatility $\hat{l}_t$ and expected returns. The average coefficient is -0.71 with a significant t-statistic of -3.89. In comparison, the relationship between expected short-run volatility $\hat{s}_t$ and the cross-section of stock returns is modest with a coefficient of 0.28 and a t-statistic of 1.65. The FM test lends support to the finding in Section 3.2 of constructing portfolios by sorting on $\hat{s}_t$ and $\hat{l}_t$. Therefore, there exists a robustly significantly negative relationship between expected long-run volatility and expected returns.

It may also be useful to explain the finding using the PT model to measure expected short-run and long-run component. While the average coefficient on expected long-run volatility $\tilde{l}_t$ is insignificant with a coefficient of -0.17. I find the time-series median of the coefficient is more significant of -0.41. This suggests that the negative relationship between the expected long-run volatility and expected returns may be downplayed by some occasional very positive coefficients in the cross-sectional regressions. The relationship may be also be related to the possibility that the PT model is mostly helpful for a group of firms with high realized idiosyncratic volatility. When sorting on $\hat{l}_t$ in Section 3.2, we have a portfolio that earns very low average returns. However, when the whole cross-section of stock returns are pooled into the FM regression, the negative relationship can be obscured by other firms in the sample .

To test this hypothesis, I perform the FM regression on a group of firms with high realized volatility (IV), the coefficient is then much larger. For the quintile 5 portfolio in Section 3.1. The average coefficient on the long-run volatility is less skewed with the mean of -1.23 and median of -2.14. The t-statistic for the coefficient is -3.83.

3.4.1 Controlling for Return Reversals

Stock returns display short-term reversals (Jegadeesh (1990) and Lehmann (1990)). Return reversal describes a phenomena that if a stock's previous month return is too high (low), it will tend to reverse the following month and earn a low (high) return. Following Huang et al. (2010), I use the returns of individual stocks in the prior month to control for return reversals. Therefore, the equation (3.7) is modified to allow for previous month's stock return

$$r^i_{t,d} = X_{i,t}\beta_t + \gamma_{l,t}\hat{l}_t + \gamma_{s,t}\hat{s}_t + \beta_{r,t-1}r^i_{t-1} + v^i_t\epsilon^i_{t,d} \tag{3.8}$$

Without previous month's stock return r^i_{t-1}, the relationship between idiosyncratic risk and expected stock returns may be negatively biased because the coefficient incorporates part of the return reversal that should have been captured by the stock returns of the previous month. Regressions including return reversals for the SL model in Table 3.7 shows that the coefficient on previous month's return is -0.06 with a t-statistic of -16.4. Even though the presence of return reversals is statistically strong, the coefficient on long-run volatility remains largely unchanged. It changes from -0.71 to -0.76 after including previous month's return. For the PT model, the evidence is also similar. Thus, the FM regression tests controlled for return reversals support a robust negative relationship between expected long-run volatility and expected returns.

3.5 Conclusion

The paper develops a new model for idiosyncratic volatility and decomposes the volatility of idiosyncratic stock return into short-run and long-run components. The results of the paper suggest that there is a robust and significant negative relationship between expected long-run volatility and expected returns. In contrast, expected short-run volatility are not found to be related to expected returns. Fu (2009) and Huang et al. (2010) propose that returns reversals can largely account for the negative relationship between idiosyncratic volatility and expected returns. The findings of this paper remain robust when stock returns in the previous month are used to control for return reversals. Finding mechanisms besides return reversals could be crucial for future research on the asset pricing implications of idiosyncratic volatility.

Since the persistent component of idiosyncratic volatility is a negative priced risk in the cross-section of stock returns, the underlying mechanism behind the phenomena may be favorably risk-based. Idiosyncratic volatility related to irrational pricing may die off quickly. Stocks with persistent high expected idiosyncratic volatilities could earn lower returns because they are exposed less to fundamental risk factors and thus earn lower risk premium. The paper suggests more work could be devoted to developing a risk-based asset pricing model to explain the negative relationship between idiosyncratic volatility and expected returns.

3.6 Tables and Figures

Table 3.1: Time Series Properties of Idiosyncratic Volatility

Panel A: Some Summary Statistics of Idiosyncratic Volatility						
Mean	Std.Dev.	Skewness	Kurtosis			
16.77	9.85	1.84	7.23			
Panel B: Autocorrelations of Idiosyncratic Volatility						
ACF(1)	ACF(2)	ACF(3)	ACF(4)	ACF(5)	ACF(10)	ACF(12)
0.33	0.26	0.23	0.18	0.16	0.10	0.09

This table summarizes the time-series statistics for idiosyncratic volatility. I first compute the statistics for each stock and the average the statistics across all stocks. The sample period is January 1963 to December 2015. The ACF stands for estimated autocorrelations at different lags. The unit of the mean and standard deviation is percentage point.

Table 3.2: Parameter Estimates for Idiosyncratic Volatility Model

Panel A: The Short and Long Run Volatility (SL) Model				
Variables	ρ_s	ρ_l	σ_s	σ_l
Mean	-0.06	0.78	0.11	0.07
Median	0.003	0.94	0.09	0.02
Panel B: The Permanent and Transitory Volatility Model				
Variables	σ_s	σ_l		
Mean	0.14	0.05		
Median	0.11	0.01		

This table summarizes the properties of parameter estimates for the short-run and long-run idiosyncratic volatility process. I first compute the parameter estimates for each stock and then construct the mean and median statistics across all stocks. The sample period is January 1963 to December 2015.

Table 3.3: Portfolios Sorted by Idiosyncratic Volatility

Rank	Mean	Std. Dev.	% Mkt Share	FF-3 alpha
1 (low)	0.92	3.77	43.1%	0.10
				[2.26]
2	0.96	4.70	31.9%	0.02
				[0.50]
3	1.00	5.80	15.4%	-0.03
				[-0.48]
4	0.73	7.23	7.2%	-0.37
				[-3.86]
5 (high)	0.08	8.53	2.5%	-1.17
				[-8.06]
5-1	-0.85	6.81		
	[-2.98]			

I form value-weighted quintile portfolios every month by sorting stocks based on idiosyncratic volatility relative to the Fama and French (1993) model. Portfolios are formed every month, based on volatility computed using daily data over the previous month. Portfolio 1 (5) is the portfolio of stocks with the lowest (highest) volatilities. The statistics in the columns labeled Mean and Std.Dev. are measured in monthly percentage terms and apply to total, not excess returns. The row $5-1$ refers to the difference in monthly returns between portfolio 5 and portfolio 1. The alpha column report Jensen's alpha with respect to the Fama-French (1993) three-factor model. Robust Newey-West (1987) t-statistics are reported in square brackets. The sample period is January 1963 to December 2015.

Table 3.4: Portfolios Sorted by the Expected Short-Run Volatility

Panel A: Short and Long-Run Volatility (SL) Model				
Rank	Mean	Std. Dev.	% Mkt Share	FF-3 alpha
1 (low)	0.90	4.55	30.7%	0.06
				[1.19]
2	0.92	4.67	17.0%	0.03
				[0.52]
3	0.95	4.96	15.3%	0.06
				[1.40]
4	1.00	4.73	15.8%	0.05
				[2.05]
5 (high)	0.93	4.79	21.2%	-0.06
				[-1.06]
5-1	0.03	2.54		
	[0.30]			
Panel B: Permanent and Transitory Volatility (PT) Model				
Rank	Mean	Std. Dev.	% Mkt Share	FF-3 alpha
1 (low)	0.94	4.64	18.5%	0.05
				[0.94]
2	0.95	4.53	21.3%	0.06
				[1.64]
3	0.90	4.52	21.5%	0.007
				[0.20]
4	0.92	4.57	21.3%	0.04
				[0.96]
5 (high)	0.80	4.86	17.3%	-0.12
				[-2.07]
5-1	-0.14	2.22		
	[-1.68]			

Portfolios are formed every month based on expected volatility of $\hat{s}_t$ or $\tilde{s}_t$. Portfolio 1 (5) is the portfolio of stocks with the lowest (highest) volatilities. The statistics in the columns labeled Mean and Std.Dev. are measured in monthly percentage terms and apply to total, not excess returns. The row $5-1$ refers to the difference in monthly returns between portfolio 5 and portfolio 1. The alpha column report Jensen's alpha with respect to the Fama-French (1993) three-factor model. Robust Newey-West (1987) t-statistics are reported in square brackets. The sample period is January 1963 to December 2015.

Table 3.5: Portfolios Sorted by the Expected Long-Run Volatility

Panel A: Short and Long-Run Volatility (SL) Model				
Rank	Mean	Std. Dev.	% Mkt Share	FF-3 alpha
1 (low)	0.87	3.86	43.6%	0.04
				[1.24]
2	1.02	4.87	34.3%	0.08
				[1.87]
3	1.12	6.11	14.8%	0.09
				[1.33]
4	0.89	7.76	5.9%	-0.3
				[-2.69]
5 (high)	0.35	9.54	1.3%	-1.03
				[-6.38]
5-1	-0.52	10.34		
	[-1.49]			
Panel B: Permanent and Transitory Volatility (PT) Model				
Rank	Mean	Std. Dev.	% Mkt Share	FF-3 alpha
1 (low)	0.91	3.80	46.7%	0.10
				[2.61]
2	1.00	4.91	32.1%	0.01
				[0.24]
3	1.05	6.31	14.0%	0.003
				[0.05]
4	0.84	8.10	5.6%	-0.30
				[-2.59]
5 (high)	0.13	9.98	1.6%	-1.17
				[-6.30]
5-1	-0.79	8.33		
	[-2.26]			

Portfolios are formed every month based on expected volatility of $\hat{l}_t$ or $\tilde{l}_t$. Portfolio 1 (5) is the portfolio of stocks with the lowest (highest) volatilities. The statistics in the columns labeled Mean and Std.Dev. are measured in monthly percentage terms and apply to total, not excess returns. The row $5 - 1$ refers to the difference in monthly returns between portfolio 5 and portfolio 1. The alpha column report Jensen's alpha with respect to the Fama-French (1993) three factor model. Robust Newey-West (1987) t-statistics are reported in square brackets. The sample period is January 1963 to December 2015.

Table 3.6: Relationship between Idiosyncratic Risk and Expected Returns: Cross-Sectional Evidence

Panel A: Short and Long-Run Volatility (SL) Model

Beta	log(ME)	log(BE/ME)	IV	$\hat{s}_t$	$\hat{l}_t$	$Ret(-1)$
0.02	-0.14	0.10				
[0.06]	[-3.85]	[2.74]				
0.15	-0.18	0.08	-0.15			
[0.63]	[-5.62]	[2.38]	[-5.12]			
-0.02	-0.15	0.11		0.28	-0.71	
[-0.07]	[-4.05]	[2.92]		[1.65]	[-3.89]	
0.01	- 0.12	0.14		0.28	-0.76	-0.06
[0.05]	[-3.43]	[3.92]		[1.68]	[-4.25]	[-16.4]

Panel B: Permanent and Transitory Volatility (PT) Model

Beta	ln(ME)	ln(BE/ME)	$\tilde{s}_t$	$\tilde{l}_t$	$Ret(-1)$
0.10	-0.16	0.10	-0.44(-0.37)	-0.17(-0.41)	
[0.47]	[-4.86]	[2.86]	[-6.25]	[-1.13]	
0.09	-0.12	0.14	-0.12(-0.06)	-0.10(-0.36)	-0.06
[0.42]	[-3.76]	[3.89]	[-1.87]	[-0.65]	[-15.6]

Stocks are assigned the post-ranking β of the size-β portfolios they are in every month. BE is the book value of common equity plus balance-sheet deferred taxes. The accounting ratios BE/ME are measured using market equity ME in December of year $t-1$. Firm size $Ln(ME)$ is measures in June of year t. In the regressions, these are values of the explanatory variables for individual stocks are matched with CRSP returns for the months from July of year t to June of year $t+1$. The gap between the accounting data and the returns ensures that the accounting data are available prior to returns. The average coefficient is the time-series average of monthly regression coefficients for July 1963 to December 2015, and the t-statistics is the average coefficient divided by its time-series standard error. The t-statistic is reported in brackets. For the PT model, the numbers in parentheses are the time series median of coefficients in the FM regression.

Table 3.7: Relationship between Idiosyncratic Risk and Expected Returns: High IV Portfolio

Panel B: Permanent and Transitory Volatility (PT) Model

Beta	ln(ME)	ln(BE/ME)	$\tilde{s}_t$	$\tilde{l}_t$	$Ret(-1)$
0.08	-0.59	0.12	-1.24(-1.30)	-1.23(-2.15)	
[0.28]	[-9.31]	[1.93]	[-5.64]	[-3.83]	
0.10	-0.46	0.20	-0.44(-0.62)	-0.74(-1.72)	-0.06
[0.36]	[-7.54]	[3.26]	[-2.03]	[-2.27]	[-13.7]

Stocks are assigned the post-ranking β of the size-β portfolios they are in every month. BE is the book value of common equity plus balance-sheet deferred taxes. The accounting ratios BE/ME are measured using market equity ME in December of year $t-1$. Firm size $Ln(ME)$ is measures in June of year t. In the regressions, these are values of the explanatory variables for individual stocks are matched with CRSP returns for the months from July of year t to June of year $t+1$. The gap between the accounting data and the returns ensures that the accounting data are available prior to returns. The average coefficient is the time-series average of monthly regression coefficients for July 1963 to December 2015, and the t-statistics is the average coefficient divided by its time-series standard error. The t-statistic is reported in brackets. For the PT model, the numbers in parentheses are the time series median of coefficients in the FM regression.

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Appendix A

Appendix for Chapter 1

A.1 Data

A.1.1 U.S Stock-Bond Data

The stock data used for U.S is the return on S&P 500. The nominal bonds and real bonds data is from Gürkaynak et al. (2007) and Gürkaynak et al. (2010).

A.1.2 U.K Stock-Bond Data

The stock data used for U.K is the return on FTSE 100 index. FTSE index began on 3 January 1984 at the base level of 1000. It is a share index of the 100 companies listed on the London Stock Exchange with the highest market capitalization. The nominal and real bonds data is from the Bank of England at :

http://www.bankofengland.co.uk/statistics/Pages/yieldcurve/archive.aspx .

A.2 Asset Pricing Moments

Equation (1.28) and (1.29) mean that the log asset return has a conditional mean that depends on s_t and is conditionally normally distributed.

$$\begin{aligned} r_{c,t+1} &= \kappa_0 + \kappa_1(A_0 + A_1' s_{t+1} + A_2'\sigma_{t+1}^2) - (A_0 + A_1' s_t + A_2'\sigma_t^2) + \Delta c_{t+1} \\ &= \kappa_0 + \kappa_1(A_0 + A_1'(G s_t + H\omega_{t+1}) + \kappa_1 A_2'[T_\sigma(\sigma_t^2 - \sigma_0^2) + \sigma_0^2 + \eta\epsilon_{\sigma,t})] \\ &- A_0 - A_1' s_t - A_2'\sigma_t^2 + \mu_c + T_c' s_t + \zeta_c' w_{t+1} \\ &= \kappa_0 + \kappa_1 A_0 - A_0 + \mu_c + (\kappa_1 A_1' G - A_1' + T_c') s_t \\ &+ (\kappa_1 A_1' H + \zeta_c') w_{t+1} + A_2'[-(I - \kappa_1 T_\sigma)(\sigma_t^2 - \sigma_0^2) + \kappa_1 \eta \epsilon_{\sigma,t}] \end{aligned} \tag{A.1}$$

Similarly,

$$\begin{aligned} r_{m,t+1} &= \kappa_{0,m} + \kappa_{1,m}(A_{0,m} + A_{1,m}' s_{t+1} + A_{2,m}'\sigma_{t+1}^2) - (A_{0,m} + A_{1,m}' s_t + A_{2,m}'\sigma_t^2) + \Delta d_{t+1} \\ &= \kappa_{0,m} + \kappa_{1,m}(A_{0,m} + A_{1,m}'(G s_t + H\omega_{t+1}) + \kappa_{1,m} A_{2,m}'[T_\sigma(\sigma_t^2 - \sigma_0^2) + \sigma_0^2 + \eta\epsilon_{\sigma,t})] \\ &- A_{0,m} - A_{1,m}' s_t - A_{2,m}'\sigma_t^2 + \mu_d + T_d' s_t + \zeta_d' w_{t+1} \\ &= \kappa_{0,m} + \kappa_{1,m} A_{0,m} - A_{0,m} + \mu_d + (\kappa_{1,m} A_{1,m}' G - A_{1,m}' + T_d') s_t \\ &+ (\kappa_{1,m} A_{1,m}' H + \zeta_d') w_{t+1} + A_{2,m}'[-(I - \kappa_{1,m} T_\sigma)(\sigma_t^2 - \sigma_0^2) + \kappa_{1,m} \eta \epsilon_{\sigma,t}] \end{aligned} \tag{A.2}$$

(A.1) have five undetermined parameters: κ_0, κ_1, A_0, and the vector A_1, A_2. We solve for the latter three using the fundamental asset pricing equation,

$$\mathbb{E}_t(M_{t,t+1}R_{t+1}) = 1$$

Since both the SDF and the return are conditionally log-normal distributed, we take log of this equation to produce

$$0 = \mathbb{E}_t(m_{t,t+1} + r_{t+1}) + \frac{1}{2}Var_t(m_{t,t+1} + r_{t+1})$$

From (1.13),

$$m_{t+1} = (1-\chi)\rho - (1-\chi)\psi\Delta c_{t+1} - \chi r_{c,t+1}$$

Therefore,

$$\begin{aligned} m_{t,t+1} = &(1-\chi)\rho - (1-\chi)\psi\mu_c - \chi[\kappa_0 + (\kappa_1 - 1)A_0 + \mu_c] - (1-\chi)\psi T_c' s_t \\ &- \chi[A_1'(\kappa_1 G - I) + T_c']s_t - (1-\chi)\psi\zeta_c'\omega_{t+1} - \chi(\kappa_1 A_1' H + \zeta_c')\omega_{t+1} \\ &- \chi A_2'[-(I - \kappa_1 T_\sigma)(\sigma_t^2 - \sigma_0^2) + \kappa_1\eta\epsilon_{\sigma,t}] \end{aligned} \tag{A.3}$$

and the innovation to $m_{t,t+1}$ follows

$$m_{t,t+1} - \mathbb{E}_t m_{t,t+1} = -\lambda_s \omega_{t+1} - \lambda_\sigma \sigma_{t+1} \tag{A.4}$$

where

$$\lambda_s = (1-\chi)\psi\zeta_c' + \chi(\kappa_1 A_1' H + \zeta_c') \tag{A.5}$$

$$\lambda_\sigma = -\chi\kappa_1 A_2'\eta \tag{A.6}$$

Also,

$$\begin{aligned}
m_{t,t+1} + r_{c,t+1} &= (1-\chi)\rho - (1-\chi)\psi\Delta c_{t+1} + (1-\chi) r_{c,t+1} \\
&= (1-\chi)\rho - (1-\chi)\psi(\mu_c + T_c' s_t + \zeta_c'\omega_{t+1}) \\
&+ (1-\chi)[\kappa_0 + \kappa_1 A_0 - A_0 + \mu_c + (\kappa_1 A_1' G - A_1' + T_c') s_t + (\kappa_1 A_1' H + \zeta_c') w_{t+1}] \\
&+ (1-\chi) A_2'[\kappa_1 T_\sigma(\sigma_t^2 - \sigma_0^2) + (\kappa_1 - 1)\sigma_0^2 + \kappa_1 \eta \epsilon_{\sigma,t}] \\
&= (1-\chi)[\rho + (1-\psi)\mu_c + \kappa_0 - (1-\kappa_1) A_0)] \\
&+ (1-\chi)[-A_1'(I - \kappa_1 G) + (1-\psi) T_c'] s_t \\
&+ (1-\chi)[(\kappa_1 A_1' H + \zeta_c' - \psi\zeta_c')\omega_{t+1} + A_2'[-(I - \kappa_1 T_\sigma)(\sigma_t^2 - \sigma_0^2) + \kappa_1 \eta \epsilon_{\sigma,t}]]
\end{aligned} \tag{A.7}$$

$$
\begin{aligned}
Var_t(m_{t,t+1} + r_{c,t+1}) = &\frac{(1-\chi)^2}{2}[\kappa_1 A_1' H + (1-\psi)\zeta_c']'\sigma_t\sigma_t'[\kappa_1 A_1' H + (1-\psi)\zeta_c'] \\
&+ \frac{(1-\chi)^2}{2}\kappa_1^2 A_2'\eta\eta' A_2
\end{aligned} \tag{A.8}
$$

Since this form of state prices and stochastic discount factor falls in the class of essentially affine models studied by Duffee (2002) et al, matching the coefficients on constant, s_t and σ_t, we have

$$
(1-\chi)[\rho + (1-\psi)\mu_c + \kappa_0 - (1-\kappa_1)A_0)] + \frac{(1-\chi)^2}{2}\kappa_1^2 A_2'\eta'\eta A_2 = 0 \tag{A.9}
$$

$$
(1-\chi)[-A_1'(I-\kappa_1 G) + (1-\psi)T_c'] = 0 \tag{A.10}
$$

$$
-(1-\chi)A_2'(I-\kappa_1 T_\sigma)\sigma_t^2 + \frac{(1-\chi)^2}{2}[\kappa_1 A_1' H + (1-\psi)\zeta_c']'\sigma_t\sigma_t'[\kappa_1 A_1' H + (1-\psi)\zeta_c'] = 0 \tag{A.11}
$$

Therefore, the loadings are given by

$$
\begin{aligned}
A_1 &= (1-\psi)(I-\kappa_1 G)^{-1}T_c' \\
A_2 &= \frac{1-\chi}{2}(I-\kappa_1 T_\sigma)^{-1}[\kappa_1 A_1' H + (1-\psi)\zeta_c']'[\kappa_1 A_1' H + (1-\psi)\zeta_c']
\end{aligned} \tag{A.12}
$$

For the stock market, which is a leverage claim to aggregate consumption,

$$
\begin{aligned}
m_{t,t+1} + r_{m,t+1} &= (1-\chi)\rho - (1-\chi)\psi\Delta c_{t+1} - \chi r_{c,t+1} + r_{d,t+1} \\
&= (1-\chi)\rho - (1-\chi)\psi(\mu_c + T_c' s_t + \zeta_c' \omega_{t+1}) \\
&- \chi[\kappa_0 + \kappa_1 A_0 - A_0 + \mu_c + (\kappa_1 A_1' G - A_1' + T_c')s_t + (\kappa_1 A_1' H + \zeta_c')w_{t+1}] \\
&- \chi A_2'[\kappa_1 T_\sigma(\sigma_t^2 - \sigma_0^2) + (\kappa_1 - 1)\sigma_0^2 + \kappa_1 \eta \epsilon_{\sigma,t}] \\
&+ \kappa_0 + \kappa_1 A_{0,m} - A_{0,m} + \mu_d + (\kappa_1 A_{1,m}' G - A_{1,m}' + T_d')s_t \\
&+ (\kappa_1 A_{1,m}' H + \zeta_d')w_{t+1} \\
&+ A_{2,m}'[-(I - \kappa_1 T_\sigma)(\sigma_t^2 - \sigma_0^2) + \kappa_1 \eta \epsilon_{\sigma,t}] \qquad \text{(A.13)} \\
&= (1-\chi)[\rho + (1-\psi)\mu_c + \kappa_0 - (1-\kappa)A_0)] \\
&+ (1-\chi)[A_1'(I - \kappa_1 G) + (1-\psi)T_c']s_t \\
&+ (1-\chi)[(\kappa_1 A_1' H + \zeta_c' - \psi\zeta_c')\omega_{t+1} + A_2'[-(I - \kappa_1 T_\sigma)(\sigma_t^2 - \sigma_0^2) + \kappa_1 \eta \epsilon_{\sigma,t}]] \\
&\qquad \text{(A.14)}
\end{aligned}
$$

and

$$
\begin{aligned}
Var_t(m_{t,t+1} + r_{c,t+1}) &= \frac{(1-\chi)^2}{2}[\kappa_1 A_1' H + (1-\psi)\zeta_c']' \sigma_t \sigma_t' [\kappa_1 A_1' H + (1-\psi)\zeta_c'] \\
&+ \frac{(1-\chi)^2}{2}\kappa_1^2 A_2' \eta' \eta A_2 \qquad \text{(A.15)}
\end{aligned}
$$

We have

$$(\kappa_{1,m}A'_{1,m}G - A'_{1,m} + T'_d) - (\kappa_1 A'_1 G - A'_1 + T'_c) = 0 \tag{A.16}$$

Since $A'_1(I - \kappa_1 G) + (1-\psi)T'_c = 0$,

$$A_{1,m} = (\phi - \psi)(I - \kappa_1 G)^{-1} T'_c$$

These solutions allows us to express the log price-dividend ratio as a function of the remaining unknown values κ_0 and κ_1 ($\kappa_{0,m}$ and $\kappa_{1,m}$ respectively for the stock market). Write this log ratio as

$$\mathcal{Q}_t = A_0(\kappa_0, \kappa_1) + A'_1(\kappa_1)s_t + A'_2(\kappa_1)\sigma_t^2$$

where the dependence of A_0 and A_1 on the unknown values is explicit. The unknown values are themselves determined by the mean of price-dividend ratio

$$\kappa_1 = \frac{\exp(\bar{\mathcal{Q}})}{1 + \exp(\bar{\mathcal{Q}})}; \quad \kappa_0 = \log(1 + \exp(\bar{\mathcal{Q}})) - \kappa_1 \bar{\mathcal{Q}}$$

and for the market portfolio

$$\kappa_{1,m} = \frac{\exp(\bar{\mathcal{Q}}_m)}{1 + \exp(\bar{\mathcal{Q}}_m)}; \quad \kappa_{0,m} = \log(1 + \exp(\bar{\mathcal{Q}}_m)) - \kappa_{1,m} \bar{\mathcal{Q}}_m$$

Plugging the price-consumption ratio and price-dividend ratio into the above formula producing

$$\kappa_1 = \frac{\exp(A_0(\kappa_0,\kappa_1) + A_1'(\kappa_1)\bar{s} + A_2'(\kappa_1)\bar{\sigma}^2)}{1+\exp(A_0(\kappa_0,\kappa_1) + A_1'(\kappa_1)\bar{\mathcal{Q}} + A_2'(\kappa_1)\bar{\sigma}^2}$$

$$\kappa_0 = 1+\exp(A_0(\kappa_0,\kappa_1) + A_1'(\kappa_1)\bar{\mathcal{Q}} + A_2'(\kappa_1)\bar{\sigma}^2$$

$$-\kappa_1\exp(A_0(\kappa_0,\kappa_1) + A_1'(\kappa_1)\bar{s} + A_2'(\kappa_1)\bar{\sigma}^2 \tag{A.17}$$

and

$$\kappa_{1,m} = \frac{\exp(A_{0,m}(\kappa_{0,m},\kappa_{0,m}) + A_{1,m}'(\kappa_{1,m})\bar{s} + A_{2,m}'(\kappa_{1,m})\bar{\sigma}^2)}{1+\exp(A_{0,m}(\kappa_{0,m},\kappa_{1,m}) + A_{1,m}'(\kappa_{1,m})\bar{\mathcal{Q}} + A_{2,m}'(\kappa_{1,m})\bar{\sigma}^2} \tag{A.18}$$

$$\kappa_{0,m} = 1+\exp(A_{0,m}(\kappa_{0,m},\kappa_{1,m}) + A_{1,m}'(\kappa_{1,m})\bar{\mathcal{Q}} + A_{2,m}'(\kappa_{1,m})\bar{\sigma}^2$$

$$-\kappa_{1,m}\exp(A_{0,m}(\kappa_{0,m},\kappa_{1,m}) + A_{1,m}'(\kappa_{1,m})\bar{s} + A_{2,m}'(\kappa_{1,m})\bar{\sigma}^2 \tag{A.19}$$

A.2.1 Bond Pricing

The bond yields are also affine in the state variables:

$$\mathcal{Y}_{t,n} = \frac{1}{n}(B_{0,n} + B_{1,n}s_t + B_{2,n}\sigma_t^2)$$

$$\mathcal{Y}_{t,n}^{\$} = \frac{1}{n}(B_{0,n}^{\$} + B_{1,n}^{\$}s_t + B_{2,n}^{\$}\sigma_t^2)$$

Plugging the above formulas into the Euler equation produces

$$\mathcal{Y}_{t,1} = B_{0,1} + B_{1,1}s_t + B_{2,1}\sigma_t^2 = -\mathbb{E}_t(m_{t,t+1}) - \frac{1}{2}Var_t(m_{t,t+1})$$
$$\mathcal{Y}_{t,1}^{\$} = B_{0,1}^{\$} + B_{1,1}^{\$}s_t + B_{2,1}^{\$}\sigma_t^2 = -\mathbb{E}_t(m_{t,t+1} + \pi_t) - \frac{1}{2}Var_t(m_{t,t+1} + \pi_t) \tag{A.20}$$

Therefore, we have

$$B_{1,1} = (1-\chi)\psi T_c' + \chi[A_1'(\kappa_1 G - I) + T_c']$$
$$B_{2,1}\sigma_t^2 = -\frac{(1-\chi)^2}{2}\psi^2\zeta_c'\sigma_t\sigma_t'\zeta_c - \frac{\chi^2}{2}(\kappa_1 A_1'H + \zeta_c')'\sigma_t\sigma_t'(\kappa_1 A_1'H + \zeta_c') + \chi A_2'[-(I - \kappa_1 T_\sigma)\sigma_t^2]$$
$$B_{0,1} = -\frac{\chi^2\kappa_1^2}{2}A_2'\eta\eta' A_2 - (1-\chi)\rho + (1-\chi)\psi\mu_c + \chi[\kappa_0 + (\kappa_1 - 1)A_0 + \mu_c] \tag{A.21}$$

and

$$B_{1,1}^{\$} = B_{1,1} + T_\pi' \tag{A.22}$$

A.2.2 Example

Suppose $A_1 = [A_{1z}, A_{1g}]'$, $s_t = [z_t, g_t]'$, $G = \begin{pmatrix} \rho_z & 0 \\ 0 & \rho_g \end{pmatrix}$, $T_c = [-(1-\rho_z)\psi_z, 1 - \psi_g(1-\rho_g)]'$, we have

$$A_{1z} = -\frac{(1-\psi)(1-\rho_z)\psi_z}{1-\kappa_1\rho_z}; \quad A_{1g} = -\frac{(1-\psi)(1-(1-\rho_g))\psi_g}{1-\kappa_1\rho_g} \tag{A.23}$$

It immediately follows that A_{1z} is negative is $\psi < 1$. In this case, the intertemporal substitution effect dominates the wealth effect. In response to lower expected consumption growth (high consumption level) today, agents sell more assets, and consequently the wealth-consumption-ratio falls. The solution coefficients $A_{2,z}$ and $A_{2,g}$ for measuring the sensitivity of price-consumption ratios to volatility fluctuations is

$$A_{2z} = \frac{(1-\gamma)[\psi_z^2(1-\sigma)^2 + (\kappa_1 A_{1z})^2]}{2(1-\kappa_1\rho_{\sigma z})}, \quad A_{2g} = \frac{(1-\gamma)[\psi_g^2(1-\sigma)^2 + (\kappa_1 A_{1g})^2]}{2(1-\kappa_1\rho_{\sigma g})}$$

The bond yields are also affine in the state variables:

$$\mathcal{Y}_{t,n} = \frac{1}{n}(B_{0,n} + B_{1,n}s_t + B_{2,n}\sigma_t^2)$$
$$\mathcal{Y}_{t,n}^{\$} = \frac{1}{n}(B_{0,n}^{\$} + B_{1,n}^{\$}s_t + B_{2,n}^{\$}\sigma_t^2)$$

The bond coefficients, which measure the sensitivity (beta) of bond prices to the aggregate risks, are pinned down by the preference and model parameters – their analytical expressions are presented in the appendix. In my model, both real and nominal yields hedge risks to trend shocks. That is, real and nominal bond prices rise and yields fall following a negative trend shock. The solution coefficients for one-period real bonds and nominal bonds are

$$B_{z,1} = -(1-\rho_z)\psi\psi_{y,z}, \quad B_{g,1} = \sigma[1-(1-\rho_g)\psi_{y,g}]$$

$$B^{\$}_{z,1} = B_{z,1} + \psi_{\pi,z}, \quad B^{\$}_{g,1} = B_{g,1} + \psi_{\pi,g}$$

Following a level shock, real interest falls as the economy is expected to revert to its long-run trend. Therefore, $B_{z,1}$ is negative. Similarly, real interest rises following trend shocks since they are positively news about future expected consumption growth.

The (nominal) return for holding a n-period bonds for one period from t to $t+1$ is

$$\begin{aligned}
r_{n,t+1} &= -(n-1)\mathcal{Y}_{t+1,n_1} + n\mathcal{Y}_{t,n} + \pi_{t+1} \\
&= B_{0,n} + B_{z,n}z_t + B_{g,n}g_t + B_{\sigma z,n}\sigma^2_{z,t} + B_{\sigma g,n}\sigma^2_{g,t} \\
&- B_{0,n-1} - B_{z,n-1}z_{t+1} - B_{g,n-1}g_{t+1} - B_{\sigma z,n-1}\sigma^2_{z,t+1} - B_{\sigma g,n-1}\sigma^2_{g,t+1} + \pi_{t+1}
\end{aligned}$$

and for nominal bonds

$$
\begin{aligned}
r^{\$}_{n,t+1} &= -(n-1)\mathcal{Y}^{\$}_{t+1,n_1} + n\mathcal{Y}^{\$}_{t,n} \\
&= B^{\$}_{0,n} + B^{\$}_{z,n} z_t + B^{\$}_{g,n} g_t + B^{\$}_{\sigma z,n}\sigma^2_{z,t} + B^{\$}_{\sigma g,n}\sigma^2_{g,t} \\
&- B^{\$}_{0,n-1} - B^{\$}_{z,n-1} z_{t+1} - B^{\$}_{g,n-1} g_{t+1} - B^{\$}_{\sigma z,n-1}\sigma^2_{z,t+1} - B^{\$}_{\sigma g,n-1}\sigma^2_{g,t+1}
\end{aligned}
$$

where

$$
B_{z,n-1} = \frac{1-\rho_z^{n-1}}{1-\rho_z} B_{z,1} \quad B_{g,n-1} = \frac{1-\rho_g^{n-1}}{1-\rho_g} B_{g,1}
$$

Since $B_{z,1} < 0$ and $B_{g,1} > 0$, we can immediately see that the relative magnitude of $\sigma_{z,t}$ and $\sigma_{g,t}$ determines the direction of the covariance between stock returns and bond returns if we focus on the first two terms in the covariance expression,

Appendix B

Appendix for Chapter 2

B.1 Empirical Part

In this section, I present the empirical evidence about the volatility of firm-level productivity.

B.1.1 Firm Level Data

The data source I use to estimate firm level productivity measure is Compustat. I use the Compustat fundamental annual data from 1962 to 2009. As it is common in the literature (Belo et al. (2014), Imrohoroglu and Tüzel (2014)), I delete observations of financial firms (SIC classification between 6000 and 6999) and regulated firms (SIC classification between 4900 and 4999). My sample for production function estimation is comprised of all remaining firms in Compustat that have positive data on sales, total assets, number of employees, gross property, plant, and equipment, depreciation, accumulated depreciation, and capital expenditures. The sample is an unbalanced panel with approximately 12,750 distinct firms spanning the years between 1962 and 2009. Following Fama and French (1992), we start our sample in 1962 since Compustat data for earlier years have a serious selection bias.

The key variables for estimating firm level productivity in our benchmark case are the firm level value added, employment, and physical capital. Firm level data is supplemented with price index for Gross Domestic Product as deflator for the value-added and price index for private fixed investment as deflator for investment and capital, both from Bureau of Economic Analysis, and national average wage index from the Social Security Administration.

Value added (y_{it}) is computed as Sales - Materials, deflated by the GDP price deflator. Sales is net sales from Compustat (SALE). Materials is measured as Total expenses minus Labor expenses. Total expenses is approximated as [Sales-Operating Income Before Depreciation and Amortization (Compustat (OIBDP))]. Labor expenses is calculated by multiplying the number of employees from the Social Security Administration. The stock of labor (l_{it}) Compustat (EMP) by average wages from the Social Security Administration. The stock of labor l_{it} is measured by the number of employees from Compustat (EMP). These steps lead to our value added definition that is proxied by Operating Income before Depreciation and Amortization + labor expenses.

Capital stock (k_{it}) is given by gross property, plant, and equipment (PPEGT) from Compustat, deflated by the price deflator for investment following the methods of Hall (1990) and Brynjolfsson and Hitt (2003). Since investment is made at various times in the past, we need to calculate the average age of capital at every year for each company and apply the appropriate deflator (assuming that investment is made all at once in year [Current Year- Age]). Average age of capital stock is calculated by dividing accumulated depreciation (Gross PPE - Net PPE, from Compustat (DPACT)) by current depreciation. The resulting capital stock is lagged by one period to measure the available capital stock at the beginning of the period.

B.1.2 Firm Level Productivity

In this paper, the production function to be estimated is given by

$$y_{it} = \beta_0 + \beta_k k_{it} + \beta_l l_{it} + \omega_{it} + \eta_{it} \tag{B.1}$$

where y_{it} is the log of value added for firm i in period t; l_{it} and k_{it} are log values of labor and capital of the firm, respectively; ω_{it} is the productivity; and η_{it} is an error term not known by the firm or the econometrician. I consider the semi-parametric procedure suggested by Olley and Pakes (1996) to estimate the parameters of this production function. This method has been recently used by Imrohoroglu and Tüzel (2014) to estimate firm level productivity. The major advantage of this approach over more traditional estimation techniques such as the ordinary least squares is its ability to control for selection and simultaneity biases and deal with the within firm serial correlation in productivity that troubles many production function estimates.

Olley and Pakes (1996) assumes that productivity w_{it}, is observed by the firm before the firm makes some of its factor input decisions, which give rise to the simultaneity problem. Labor, l_{it} is the only variable input, i.e, its value can be affected by the current productivity, ω_{it}. The other input, k_{it}, is a fixed input at time t, and its value is only affected by the conditional distribution of w_{it} at time $t-1$. Consequently, w_{it} is a state variable that affects firms' decision making where firms that observe a positive probability shock in period t will invest more in capital, i_{it}, and hire more labor, l_{it}, in that period. The solution to the firm's optimization problem results in the equation for i_{it}:

$$i_{it} = i(\omega_{it}, k_{it}) \tag{B.2}$$

where both i and j are strictly increasing in ω. The inversion of the equations yield:

$$\omega_{it} = h(i_{it}, k_{it}) \tag{B.3}$$

where h is strictly increasing in i_{it}. We can define $\phi_{it} = \beta_0 + \beta_k k_{it} + h(i_{it}, k_{it})$. Substituting ϕ_{it} into (B.1) yields

$$y_{it} = \beta_l l_{it} + \phi_{it} + \eta_{it} \tag{B.4}$$

where we approximate ϕ_{it} with a second order polynomial series in capital and investment. This first stage estimation results in an estimate for $\hat{\beta}_l$ that controls for the simultaneity problem. In the second stage, consider the expectation of $y_{i,t+1} - \hat{\beta}_l l_{i,t+1}$ on information at time t and survival of the firm:

$$\mathbb{E}_t(y_{i,t+1} - \hat{\beta}_l l_{i,t+1}) = \beta_0 + \beta_k k_{i,t+1} + \mathbb{E}_t(\omega_{it+1}|\omega_{it}, survival) \tag{B.5}$$

$$= \beta_0 + \beta_k k_{i,t+1} + g(\omega_{it}, \hat{P}_{\text{survival},t}) \tag{B.6}$$

The survival probability is estimated via a probit of a survival indicator variable on a polynomial expression containing capital and investment. We fit the following equation by nonlinear least squares:

$$y_{i,t+1} - \hat{\beta}_l l_{i,t+1} = \beta_k k_{i,t+1} + \rho\omega_{it} + \tau\hat{P}_{\text{survival},t} + \eta_{i,t+1} \tag{B.7}$$

where ω_{it} is given by $w_{it} = \phi_{it} - \beta_0 - \beta_k k_{it}$ and is assumed to follow an AR(1) process. At the end of this stage, $\hat{\beta}_l$ and $\hat{\beta}_k$ are estimated. Finally, productivity is measured by:

$$p_{it} = \exp(y_{it} - \beta_0 - \hat{\beta}_l l_{it} - \hat{\beta}_k k_{it}) \tag{B.8}$$

The estimates for the production function are summarized in the Table A.1. The production function estimated display nearly constant returns to scale and are insensitive to the samples used. The properties of firm level productivities are summarized in Table A.2

Table A.1: Estimates for Production Function Parameters

Sample	Obs	β_k	β_l	$\beta_k + \beta_l$
Compustat	103707	0.228	0.746	0.9740
CRSP/Compustat	84130	0.2614	0.7319	0.9933

Notes: I use both the Compustat and the Compustat/CRSP merged dataset. The production function estimates displayed are stable and insensitive to the sample chosen. The production function displays constant returns to scale.

Table A.2: Summary Statistics for Firm Level Productivity

Sample	Obs	Mean	Std.Dev.	Skewness	Kurtosis	Percentiles		
						0.05	0.50	0.95
Compustat	103707	0.068	0.043	37.70	3235.597	0.036	0.063	0.113
CRSP/Compustat	84130	0.072	0.049	42.82	3625.37	0.039	0.066	0.120

B.2 Solution of the Model When Productivity follows Geometric Brownian Motion

This section develops an analytical solution of the model when productivity follows a Geometric Brownian motion. The analysis draws on Bertola and Caballero (1994) and Stokey (2008).

B.2.1 Firm Investment Problem

This section provides details to characterize the firm investment rule in steady state. First, I reduce the dimensionality of the optimization problem by the homogeneity property of value function. By the virtue of this simplification, I could obtain closed form solution for firm investment problem when the idiosyncratic productivity follows. The aggregated investment function can then be computed.

B.2.1.1 Exploiting Homogeneity

In my model setup, the profit function is homogeneous of degree one,

$$\Pi(K, Z) = Z\pi(K/Z)$$

where $\pi(\hat{K}) \equiv \Pi(\hat{K}, 1)$. Also define the piecewise linear function

$$\rho(I) = \begin{cases} I, \ I \geq 0 \\ 0, \ I < 0 \end{cases}$$

The firm's problem can then be written as

$$V(K_t, Z_t) = \max_{\{I(t)\}} \mathbb{E}_t \left[\int_0^\infty e^{-ru} \{[Z\pi(K/Z) - K[\rho(I/K)]\} du] \right] \tag{B.9}$$

subject to

$$\begin{aligned} \frac{dK}{K} &= (\frac{I}{K} - \delta)dt \\ dZ/Z &= \mu dt + \sigma dW \end{aligned} \tag{B.10}$$

The associated HJB equation is

$$\begin{aligned} rV =& Z\pi(K/Z) - \delta K V_K + \mu V_Z + \frac{1}{2}\sigma^2 V_{ZZ} \\ &+ \max_I \{V_K I - K\rho(I/K)\} \end{aligned} \tag{B.11}$$

where V and its derivatives are evaluated at $(K/Z, 1)$. This second order PDE can be written as an ODE by exploiting homogeneity. The return function and constraints in (B.9) and (B.10) are homogeneous of degree one in (K, Z, I). Hence V is homogeneous of degree one in (K, Z), and the optimal policy is homogeneous in the sense that if the stochastic process I is optimal for the initial conditions (K_t, Z_t), then for any $\lambda > 0$ the process λI^* is optimal for the initial conditions $(\lambda K_t, \lambda Z_t)$. Define the ratios $\mathcal{K} \equiv K/Z$ and $\mathcal{I} \equiv I/K$, and the intensive form of the value function $v(\mathcal{K}) \equiv V(\mathcal{K}, 1)$, $\mathcal{K} \geq 0$. Then

$$V(K, Z) = Zv(K/Z), \qquad \text{all } K, Z$$

Thus,

$$V_K = v', \quad V_Z = v - \mathcal{K}v', \quad V_{ZZ} = \mathcal{K}^2 \frac{1}{Z} v''$$

Substituting for V and its derivatives in (B.11) gives the HJB equation in the intensive form

$$(r - \mu)v = \pi(\mathcal{K}) - (\delta + \mu)\mathcal{K}v' + \frac{1}{2}\sigma^2 \mathcal{K}^2 v'' + \mathcal{K} \max_{\mathcal{I} \geq 0} [v'\mathcal{I} - \rho(\mathcal{I})] \tag{B.12}$$

The coefficient on v in the normalized HJB equation is $r - \mu(Z)$. Since the investment problem is a special case in which the investment is defined by a threshold, we could obtain the result below. If $K < b(Z)$, the firm makes a discrete investment of size $b(Z) - K$, so below the threshold the value function is

$$V(K, Z) = V[b(Z), Z] + b(Z) - K, \quad K < b(Z)$$

Therefore, investment is just sufficient to keep K from falling below $b(Z)$. The region above $b(Z)$ is the inaction region. In this region the value function satisfies the HJB equation in the intensive form

$$(r-\mu)v = \pi(\mathcal{K}) - (\delta+\mu)\mathcal{K}v' + \frac{1}{2}\sigma^2\mathcal{K}^2 v'' \tag{B.13}$$

In the region where the firm makes discrete investments

$$v(\mathcal{K}) = v(b^*) + (b^* - \mathcal{K}), \quad \mathcal{K} < b^* \tag{B.14}$$

The optimal threshold has the form $b(Z) = b^* Z$ where the constant b^* must be determined. Thus, by exploiting the homogeneity property of the value function, I reduce the second order partial differential equation (PDE) of (2.10) into a normalized HJB equation in the form of an ordinary differential equation (ODE).

B.3 Closed Form Solutions under Geometric Brownian Motion Productivity Process

B.3.1 Solving the ODE

When $\Pi(K, Z) = K^{\alpha} Z^{1-\alpha}$ and $dZ/Z = \mu dt + \sigma dW_t$, the intensive form of the HJB equation is

$$(r - \mu)v = k^{\alpha} - (\delta + \mu)kv' + \frac{1}{2}\sigma^2 k^2 v'', \quad k > b^* \tag{B.15}$$

The normalized HJB equation (B.15) is a standard second order linear nonhomogeneous differential equation. All solutions have the form

$$v(\mathcal{K}) = v_p(\mathcal{K}) + \alpha_1 h_1(\mathcal{K}) + \alpha_2 h_2(\mathcal{K}), \quad \mathcal{K} > b^*$$

where $v_p(\mathcal{K})$ is any particular solution, $h_i(\mathcal{K})$, $i = 1, 2$, are homogeneous solutions. It is easy to verify that a particular solution has the form of

$$v_p(\mathcal{K}) = \frac{1}{\eta}\mathcal{K}^{\alpha}$$

.

The homogeneous solutions are $h_i(\mathcal{K}) = \mathcal{K}^{R_i}$, $i = 1, 2$, where R_1 and R_2 are the roots of the quadratic

$$0 = (r - \mu) + (\delta + \mu)R - \frac{1}{2}\sigma^2 R(R - 1)$$

The assumption $r > \mu$ insures the roots are real and of opposite sign. Label them $R_1 < 0 < R_2$. Therefore, all solutions can be written as

$$v(\mathcal{K}) = \frac{1}{\eta}\mathcal{K}^\alpha + \alpha_1 \mathcal{K}^{R_1} + \alpha_2 \mathcal{K}^{R_2}, \quad k > b^*$$

where the constants α_1 and α_2 must be determined. Since there is no upper threshold,

$$\lim_{\mathcal{K}\to\infty} \left(v(\mathcal{K}) - \frac{1}{\eta}\mathcal{K}^\alpha = 0 \right)$$

reflecting the fact that as $\mathcal{K} \to \infty$, the time until investment is positive becomes arbitrarily long, with probability arbitrarily close to one. Since $R_1 < 0 < R_2$, this condition holds if and only if $\alpha_2 = 0$. Let R (without a subscript) denote the negative root, so the value function has the form

$$v(\mathcal{K}) = \begin{cases} \mathcal{K}^\alpha/\eta + a_1 \mathcal{K}^R, & \mathcal{K} \geq b^* \\ v(b^*) - (b^* - \mathcal{K}) & 0 \leq \mathcal{K} < b^* \end{cases}$$

where R satisfies

$$R \equiv \frac{1}{\sigma^2}(m - D)$$
$$D \equiv \left[m^2 + 2\sigma^2(r - \mu)\right]^{1/2}$$
$$m \equiv \delta + \mu + \frac{1}{2}\sigma^2$$

It remains to determine a_1 and b^*. The smooth pasting condition, $\lim_{k \downarrow b^*} v'(\mathcal{K}) = P$ suggests that

$$a_1 = \frac{1}{R}\left[P(b^*)^{1-R} - \frac{\alpha}{\eta}(b^*)^{\alpha - R}\right]$$

The super contact condition $\lim_{k \downarrow b^*} v''(\mathcal{K}) = 0$ requires that

$$b^* = A^{1/(\alpha-1)}$$

where

$$A \equiv \frac{\eta}{\alpha}\frac{1-R}{\alpha - R}$$

Write A, η, m, D, and R as functions of σ^2 and use to find that

$$\frac{\partial A/\partial\sigma^2}{A} = \frac{\partial\eta/\partial\sigma^2}{\eta} + \frac{(1-\alpha)\partial R/\partial\sigma^2}{(1-R)(\alpha-R)}$$

Clearly $\partial\eta/\partial\sigma^2 > 0$, so the first term is positive. The second term is also positive if

$\partial R/\partial\sigma^2 > 0$ It can be shown that

$$\frac{\partial R}{\partial \sigma^2} = \frac{D-m}{2D\sigma^2}\left(\frac{D-m}{\sigma^2}+1\right) > 0$$

Thus when investment is irreversible a higher variance σ^2 leads to a lower investment threshold b^*. That is the optimal policy allows the ratio of the capital stock to demand to fall farther before triggering positive investment. In irreversible case greater uncertainty reduces investment.

Proposition 3 *In the steady state of the model, the value function associated with* (2.8) *has the form*

$$V(K,Z) = Zv(\mathcal{K})$$
$$v(\mathcal{K}) = v(b^*) + (b^* - \mathcal{K}), \quad \mathcal{K} < b^*$$
$$v(\mathcal{K}) = v_p(\mathcal{K}) + \alpha_1 h_1(\mathcal{K}), \quad \mathcal{K} \geq b^*$$

where $\mathcal{K} \equiv K/Z$, $v(\mathcal{K})) \equiv V(\mathcal{K},1)$, $v_p(\mathcal{K}) = \frac{1}{\eta}\mathcal{K}^\alpha$ *and* $h_1(\mathcal{K}) = \mathcal{K}^{R_1}$. *The constants* *are defined as*

$$
\begin{aligned}
\eta &\equiv (r-\mu) + \alpha(\delta+\mu) - \alpha(\alpha-1)\frac{1}{2}\sigma^2 \\
a_1 &\equiv \frac{1}{R}\left[P(b^*)^{1-R} - \frac{\alpha}{\eta}(b^*)^{\alpha-R}\right] \\
A &\equiv \frac{\eta}{\alpha}\frac{1-R}{\alpha-R} \\
b^* &\equiv (AP)^{1/(\alpha-1)} \\
R &\equiv \frac{m-D}{\sigma^2} \\
D &\equiv [m^2 + 2\sigma^2(r-\mu)]^{1/2} \\
m &\equiv \sigma + \mu + \frac{1}{2}\sigma^2
\end{aligned}
$$

The investment threshold has the form $b(Z) = b^*Z$. The investment function therefore follows

$$
dL_t = \begin{cases} 0 & \text{if } K > b(Z) \\ b(Z) - K = b^*Z - K & \text{if } K \leq b(Z) \end{cases}
$$

The main insight from the above formula is when investment is irreversible a higher variance σ^2 leads to a lower investment threshold b^*. That is because the optimal policy allows the ratio of the capital stock to demand to fall farther before triggering positive investment. In irreversible case, greater uncertainty reduces investment.

B.3.2 Cross-sectional Distribution of Firm Capital Growth Rates

To study the aggregate investment, it is necessary to track the whole cross-sectional density of firm capital growth. It is useful to introduce a few notations to characterize the cross-sectional distribution of investments. Let $k_{i,t} \equiv \log(K_{i,t})$ denote the log capital for firm i. I use $k_t^* = \log b^* + \log z$ to denote the log of "desired" capital if were no irreversibility constraint at time t. Also define $s_t \equiv k_t - k_t^*$ as the difference fo the log capital stock from the log "desired" capital stock. By Ito's lemma

$$dk_t^* = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t$$

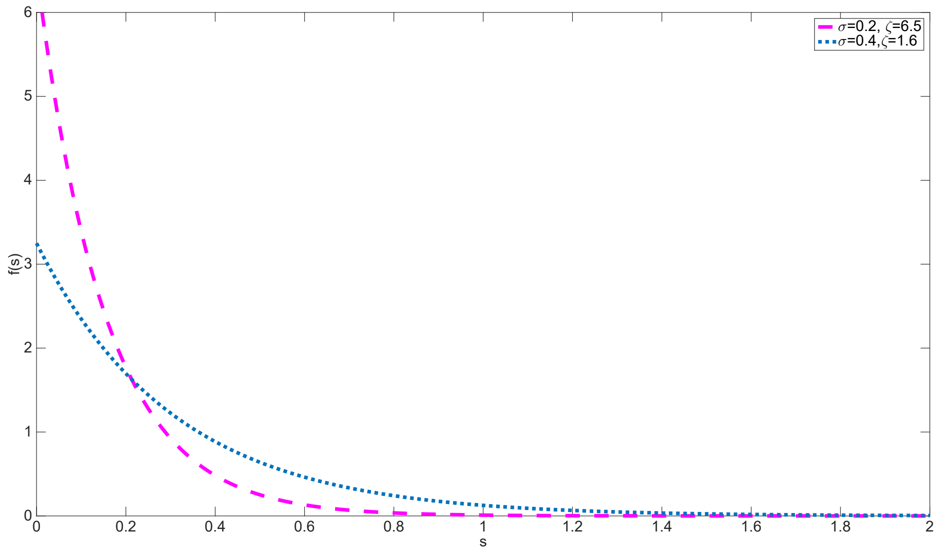
$$ds_t = dk_t - dk_t^* = \begin{cases} 0 & \text{if } dL_t > 0 \\ -\delta dt - dk_t^* & \text{if } dL_t = 0 \end{cases}$$

Let $f(s,t)$ denotes the density function for $s_{i,t}$. Since there is no aggregate shock in the economy, the cross-sectional density has settled into a steady state. Since the number of firms is large, the steady state cross-sectional density corresponds to the ergodic density of a single s_i. As each s_i behaves as a Brownian motion regulated at 0, with standard deviation σ and drift $\nu \equiv -(\mu - \sigma^2/2 + \delta)$, the steady state density is exponential (see appendix):

$$f(s) = \zeta e^{-\zeta s} \quad s \geq 0, \quad \text{where } \zeta \equiv -\frac{2\nu}{\sigma^2} \tag{B.16}$$

Figure B.1 plots the steady state densities for two positive values of σ. With positive depreciation $\delta > 0$ and a secular tendency for desired investment to be positive ($\mu - \sigma^2/2 > 0$), we have $\zeta > 0$ and every individual tends to drift towards the investment point, where $s_t = 0$. Hence, in steady state more units are found in the neighborhood of $s = 0$ than farther from it. Because of the presence of the irreversibility constraint, the high volatility of productivity shocks makes firm's investment risky and therefore reduce the incentive to invest. The larger is the volatility of shocks, the smaller is the measure of units investing at any point in time and thus the smoother is the slope of the cross-sectional density.

Figure B.1: The Cross Sectional Density of Firm Investment Rate



This figures plots the cross sectional density for firm investment. The pink dashed line corresponds to smaller idiosyncratic productivity shock $\sigma = 0.2$, while the blue dotted line corresponds to larger idiosyncratic productivity shock $\sigma = 0.4$. As the volatility of idiosyncratic productivity shock becomes larger, more firms are constrained in the inaction region. Therefore, the cross sectional density is more spread out.

B.3.3 Stationary Distribution

Let $f(s,t)$ denotes the probability density of the process s_t with stochastic differential equation

$$ds(t) = \nu dt + \sigma dW(t), \quad \sigma > 0$$

where $\{W(t)\}$ is a standard Wiener process, and let $\{s\}$ be reflected at 0 and $\bar{s} > 0$. The function $f(s,t)$ can be derived by solving the forward Kolmogorov equation

$$\partial_t f(s,t) = \frac{1}{2}\sigma^2 \partial_{ss} f(s,t) - \nu \partial_s f(s,t) \tag{B.17}$$

with boundary conditions

$$\begin{aligned} \frac{1}{2}\sigma^2 \partial_s f(0,t) &= \nu f(0,t), \forall t \\ \frac{1}{2}\sigma^2 \partial_s f(s,t) &= \nu f(s,t), \forall t \end{aligned}$$

and given initial condition

$$f(s,0) = \bar{g}(s), \quad \int_0^S \bar{g}(s) ds = 1$$

Separating the variables, we write $f(s,t) = g(s)h(t)$ and obtain a couple of ordinary differential equations. In the t direction,

$$h'(t) + \lambda h(t) = 0$$

has general solution $h(t) = Ae^{-\lambda t}$, A is a constant of integration. In the s direction,

$$\begin{aligned} g''(s) + \zeta g'(s) - \lambda \frac{\zeta}{\nu} g(s) &= 0 \\ g'(0) &= -\zeta g(0) \\ g'(S) &= -\zeta g(S) \end{aligned}$$

where $\zeta = -2\nu/\sigma^2, \zeta > 0$. It defines a Sturn-Liouville problem with characteristic equation

$$a^2 + \zeta a - \frac{\lambda}{\nu}\zeta = 0$$

If $\lambda \leq -\zeta\nu/4 = \zeta^2\sigma^2/8$, the roots are real and solutions taken the general form

$$g(s) = A_1 e^{a_1 s} + A_2 e^{a_2 s} \tag{B.18}$$

Solutions in this form need be considered only if they can satisfy the boundary condition with A_1 and/or A_2 different from 0. There exist solutions if $\lambda = 0$, which corresponding to the steady state equilibrium.

$$g(s) = \zeta e^{-\zeta s} \tag{B.19}$$

Curriculum Vitae

Yunting Liu was born in Wuhan, China on August 23, 1989. He attended the Wuhan University in China from which he obtained B.S. in Mathematics and B.A. in Financial Economics in 2011. In the same year, he enrolled in the Ph.D. in Economics program at the Johns Hopkins University in Baltimore, MD. He obtained a Master of Arts degree in 2014 and completed a Doctor of Philosophy in August 2017. He will be joining Peking University as an assistant professor in September 2017.